Optimal Approximation by sk-Splines on the Torus

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Key words and phrases: spline, torus, interpolation, approximation, multiplier, Sobolev spaces

MSC2010 classification numbers: 41A15, 41A25, 41A65

Abstract

Fixed a continuous kernel K on the d-dimensional torus, we consider a generalization of the univariate sk-spline to the torus, associated with the kernel K. It is proved an estimate which provides the rate of convergence of a given function by its interpolating sk-splines, in the norm of L^q for functions of the type $f = K * \varphi$ where $\varphi \in L^p$ and $1 \leq p \leq 2 \leq q \leq \infty,\ 1/p - 1/q \geq 1/2$. The rate of convergence is obtained for functions f in Sobolev classes and this rate gives optimal error estimate of the same order as best trigonometric approximation, in a special case.

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1 Introduction

The sk-splines are a natural generalization of the polynomial splines and of the $\mathcal{L}$-splines of Micchelli [11]. The sk-splines were introduced and their basic theory developed by A. K. Kushpel. The latest results about convergence of sk-splines on the circle in the spaces L^q, were obtained by Kushpel in [6, 7]. For an overview of approximation by sk-splines see [11].

In Section 3 we introduce the concept of sk-spline on the torus $\mathbb{T}^d$ and we show some basic results. In Section 4 we define the fundamental sk-spline, interpolating sk-splines and we find conditions for the existence and uniqueness of interpolating sk-splines of a given function. We show that the interpolating sk-spline can be obtained from the fundamental sk-spline. In Section 5 we prove Theorem 5.7 which provides the rate of convergence of a given function by its interpolating sk-splines. The rate of convergence is given in the norm of $L^q(\mathbb{T}^d)$ for functions of the type $f = K * \varphi$ where $\varphi \in L^p(\mathbb{T}^d)$ and K is a fixed kernel, for $1 \le p \le 2 \le q \le \infty$ where $p^{-1} - q^{-1} \ge 2^{-1}$. The most important result for our applications is the Corollary 5.9.

Consider fixed a kernel K. Given $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$ let $\mathbf{x_k} = (x_{k_1}, \dots, x_{k_d}),\ x_{k_l} = \pi k_l/n_l$. We denote by $sk_{\mathbf{n}}(f, \cdot)$ the unique interpolating sk-spline of a function f with set of knots and interpolating points $\Lambda_{\mathbf{n}} = \{\mathbf{x_k} : 0 \le k_l \le 2n_l - 1, 1 \le l \le d\}$. In the last section we prove that if $\gamma \in \mathbb{R},\ \gamma > d$,

$$K(\mathbf{x}) = \sum_{\mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} |\mathbf{l}|^{-\gamma} e^{i\mathbf{l}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{T}^d, \tag{1}$$

where $|\cdot|$ is the norm $|\cdot|_2$ or $|\cdot|_\infty$ on $\mathbb{R}^d$, $\mathbf{n} = (n, \dots, n) \in \mathbb{N}^d$, $1 \le p \le 2 \le q \le \infty$, with $1/p - 1/q \ge 1/2$, then there is a positive constant $C_{p,q}$, independent of $n \in \mathbb{N}$, such that

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le C_{p,q} n^{-\gamma + d(1/p - 1/q)}. \tag{2}$$

The set $K * U_p = \{K * \varphi : \varphi \in U_p\}$, where U_p is the unit ball of $L^p(\mathbb{T}^d)$, is a Sobolev class on the torus $\mathbb{T}^d$.

It follows from [10] and [13] that for $1 \leq p \leq 2 \leq q < \infty$, the $(2n)^d$-width of Kolmogorov of $K * U_p$ verifies

$$d_{(2n)^d}(K * U_p, L^q) \asymp n^{-\gamma + d(1/p - 1/2)}. \tag{3}$$

For $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^d$, the dimension of the space $SK(\Lambda_{\mathbf{n}})$ of interpolating sk-splines on $\Lambda_{\mathbf{n}}$ is $(2n)^d$. Comparing (2) and (3) we can see that the rate of convergence by interpolating sk-splines is as good as the rate of convergence by subspaces of trigonometric polynomials of the dimension of $SK(\Lambda_{\mathbf{n}})$ on Sobolev classes, when $p = 1$ and $q = 2$, that is, the rate of convergence is optimal in the sense of n-widths. The construction of the optimal interpolating sk-splines is given in Theorem 4.5. $SK(\Lambda_{\mathbf{n}})$ is an optimal subspace for the Kolmogorov $(2n)^d$-width of the Sobolev class $K * U_1$ in L^2.

In [4, 5] the convergence of sk-splines for functions in anisotropic Sobolev classes on the torus was studied. These studies were improved in [3, 2]. The best result was obtained in [2]. It was proved an almost optimal estimate, in the sense of best approximation by trigonometric polynomials, for functions in Sobolev classes, by sk-splines, optimal up to a logarithmic factor. In [9] it was obtained a similar result for the case $p = q = 1$.

2 Preliminaries

If (a_n) and (b_n) are sequences, we write $a_n \gg b_n$ to indicate that there is a constant $C_1 > 0$ such that $a_n \geq C_1 b_n$ for all $n \in \mathbb{N}$ and we write $a_n \ll b_n$ to indicate that there is a constant $C_2 > 0$ such that $a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$. We write $a_n \asymp b_n$ to indicate that $a_n \ll b_n$ and $a_n \gg b_n$.

The $d-$dimensional torus $\mathbb{T}^d$ is defined as the product of d copies of the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, or $\mathbb{T}^d = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \times \cdots \times \mathbb{R}/2\pi\mathbb{Z}$. We can identify $\mathbb{T}^d$ with the $d-$dimensional cube $[-\pi,\pi]^d$ and also with the cartesian product $S^1\times\cdots\times S^1$, of d times the unitary circle $S^1=\{e^{it}:t\in[-\pi,\pi]\}$. We will consider $\mathbb{T}^d$ endowed with the normalized Lebesgue measure $d\nu(\mathbf{x})=\left(1/(2\pi)^d\right)dx_1dx_2\cdots dx_d$, where $(1/2\pi)\,dt$ is the normalized Lebesgue measure on S^1.

For $\mathbf{l}=(l_1,\dots,l_d)$, $\mathbf{k}=(k_1,\dots,k_d)$, $\mathbf{j}=(j_1,\dots,j_d)\in\mathbb{Z}^d$ and $\mathbf{x}=(x_1,\dots,x_d)$, $\mathbf{y}=(y_1,\dots,y_d)\in\mathbb{R}^d$, we denote $\mathbf{x}\cdot\mathbf{y}=x_1y_1+\cdots+x_dy_d$; $\mathbf{lk}=(l_1k_1,\dots,l_dk_d)$; $\mathbf{l}\equiv\mathbf{k}\ \mathrm{mod}(\mathbf{j})$ if there is $\mathbf{p}\in\mathbb{Z}^d$ such that $\mathbf{l}-\mathbf{k}=\mathbf{pj}$; $\mathbf{0}=(0,0,\dots,0)$; $\mathbf{1}=(1,1,\dots,1)$; $|\mathbf{x}|_p=(|x_1|^p+|x_2|^p+\cdots+|x_d|^p)^{1/p}$ for $1\le p<\infty$; $|\mathbf{x}|_\infty=\max_{1\le j\le d}|x_j|$.

In this paper we consider a arbitrary norm $\mathbf{x}\to|\mathbf{x}|$ on $\mathbb{R}^d$ and we denote by $|\mathbf{l}|$ the norm of the element $\mathbf{l}\in\mathbb{Z}^d$.

We denote by $L^p=L^p(\mathbb{T}^d),\ 1\le p\le\infty$, the vector space of all mensurable functions f defined on $\mathbb{T}^d$ and with values in $\mathbb{C}$, satisfying

$$
\begin{aligned}
||f||_p &= \left(\int_{\mathbb{T}^d}|f(\mathbf{x})|^p d\nu(\mathbf{x})\right)^{1/p}<\infty,\ 1\le p<\infty,\\
||f||_\infty &= \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{T}^d}|f(\mathbf{x})|<\infty.
\end{aligned}
$$

We write $U_p=\{f\in L^p(\mathbb{T}^d):||f||_p\le 1\}$.

Given $f\in L^1(\mathbb{T}^d)$ we define the Fourier series of the function f by

$$\sum_{\mathbf{m}\in\mathbb{Z}^d}\widehat{f}(\mathbf{m})e^{i\mathbf{m}\cdot\mathbf{x}}.$$

where

$$\widehat{f}(\mathbf{m})=\int_{\mathbb{T}^d}f(\mathbf{x})e^{-i\mathbf{m}\cdot\mathbf{x}}d\nu(\mathbf{x}).$$

The convolution product of two functions f and g in $L^1(\mathbb{T}^d)$, denoted by $f*g$, is defined by

$$f*g(\mathbf{x})=\int_{\mathbb{T}^d}f(\mathbf{x}-\mathbf{y})g(\mathbf{y})d\nu(\mathbf{y}).$$

If $1 \leq p, q \leq \infty,\ f \in L^q(\mathbb{T}^d)$ and $g \in L^p(\mathbb{T}^d)$, then the Young Inequality says that $f * g \in L^s(\mathbb{T}^d)$, where $1/s = 1/p + 1/q - 1$, and

$$||f * g||_s \leq ||f||_q ||g||_p.$$

Let $(a_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$ be a sequence of real numbers such that $a_{\mathbf{l}} = a_{-\mathbf{l}}$ for every $\mathbf{l} \in \mathbb{Z}^d$ and

$$\sum_{\mathbf{l}\in\mathbb{Z}^{\mathbf{d}}} |a_{\mathbf{l}}| < \infty.$$

Consider the kernel $K(\mathbf{x})$ given by

$$K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}}.$$

We have that K is a real function, continuous and even. We consider the convolution operator defined for $f \in L^1(\mathbb{T}^d)$ by

$$Tf(x) = K * f(x),\ x \in \mathbb{T}^d.$$

T is a bounded linear operator from $L^p(\mathbb{T}^d)$ to $L^q(\mathbb{T}^d)$, for $1 \leq p, q \leq \infty$. For $f \in L^1(\mathbb{T}^d)$ we have

$$Tf(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} \hat{f}(\mathbf{l}) e^{i\mathbf{l}\cdot\mathbf{x}},$$

and the norm of T as an operator from L^p to L^q is the norm of T as an operator from $L^{p'}$ to $L^{q'}$, that is $||T||_{p,q} = ||T||_{q',p'}$, for every $p, q \in \mathbb{R}$, $1 \leq p, q \leq \infty$, where p' and q' satisfy $1/p + 1/p' = 1/q + 1/q' = 1$. We denote

$$K * U_p = \{K * f : f \in U_p\}.$$

For $l, N \in \mathbb{N}$ we define

$$A_l = \{\mathbf{k} \in \mathbb{Z}^d : |\mathbf{k}|_2 \leq l\}, \quad A_l^* = \{\mathbf{k} \in \mathbb{Z}^d : |\mathbf{k}|_\infty \leq l\},$$

$$\mathcal{H}_l = [e^{i\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in A_l \setminus A_{l-1}],\ \mathcal{H}_l^* = [e^{i\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in A_l^* \setminus A_{l-1}^*],$$

where $A_{-1} = A^*_{-1} = \varnothing$ and $[f_j : j \in \Gamma]$ denotes the vector space generated by the functions $f_j : \mathbb{T}^d \longrightarrow \mathbb{C}$, with j in the set of indexes Γ. We denote by $\mathcal{H}$ the vector space generated by the family $\{e^{i\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in \mathbb{Z}^d\}$ which is dense in $L^p(\mathbb{T}^d)$ for $1 \leq p < \infty$. Then by [1] and [12], there are positive constants C_1, C_2 and C_3 satisfying

$$\frac{2\pi^{d/2}}{\Gamma(d/2)} l^{d-1} - C_2 l^{d-2} \leq \dim \mathcal{H}_l \leq \frac{2\pi^{d/2}}{\Gamma(d/2)} l^{d-1} + C_1 l^{d-2}.$$

It is easily to verify that there is a positive constant C such that for every $l, N \in \mathbb{N}$,

$$\dim \mathcal{H}_l{}^* = (2l+1)^d - (2(l-1)+1)^d \asymp l^{d-1}.$$

In particular $\dim \mathcal{H}_l \asymp \dim \mathcal{H}_l^* \asymp l^{d-1}$.

Consider a function $\lambda : [0, \infty) \to \mathbb{R}$ and let $\mathbf{x} \to |\mathbf{x}|$ be a norm on $\mathbb{R}^d$. For each $\mathbf{k} \in \mathbb{Z}^d$ we define $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$. We denote by Λ the linear operator defined for $\varphi \in \mathcal{H}$ by

$$\Lambda\varphi = \sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{\mathbf{k}} \widehat{\varphi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\lambda_{\mathbf{k}} \in \mathbb{C}$, and $1 \leq p, q \leq \infty$. If for any $\varphi \in L^p(\mathbb{T}^d)$ there is a function $f = \Lambda\varphi \in L^q(\mathbb{T}^d)$ with formal Fourier expansion given by

$$f \sim \sum_{\mathbf{k}\in\mathbb{Z}^d} \lambda_{\mathbf{k}} \widehat{\varphi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

such that $||\Lambda||_{p,q} = \sup\{||\Lambda\varphi||_q : \varphi \in U_p\} < \infty$, we say that Λ is a bounded multiplier operator from $L^p(\mathbb{T}^d)$ into $L^q(\mathbb{T}^d)$, with norm $||\Lambda||_{p,q}$.

3 Basic results

Let $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ we denote $x_{k_l} = \pi k_l / n_l$, $1 \leq l \leq d$ and $\mathbf{x_k} = (x_{k_1}, \ldots, x_{k_d})$. We also denote

$$\Omega_{\mathbf{n}} = \{\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d : \ 0 \leq j_l \leq 2n_l - 1, \ 1 \leq l \leq d\},$$

$$\Lambda_{\mathbf{n}} = \{\mathbf{x_k} : \mathbf{k} \in \Omega_{\mathbf{n}}\}, \qquad N = \#\Omega_{\mathbf{n}} = \#\Lambda_{\mathbf{n}} = 2^d n_1 n_2 \cdots n_d.$$

The real vector space of all continuous functions $f : \mathbb{T}^d \to \mathbb{R}$ endowed with the norm of the uniform convergence will be denoted by $C(\mathbb{T}^d)$.

For a fixed kernel $K \in C(\mathbb{T}^d)$, a sk-spline on $\Lambda_{\mathbf{n}}$ is a function represented in the form

$$sk_{\mathbf{n}}(\mathbf{x}) = c + \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{k}} K(\mathbf{x} - \mathbf{x_k}),$$

where the coefficients c, $c_{\mathbf{k}} \in \mathbb{R}$, $\mathbf{k} \in \Omega_{\mathbf{n}}$, satisfy the condition

$$\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{k}} = 0.$$

The points $\mathbf{x_k}$ are the knots of the sk-spline $sk_{\mathbf{n}}(\mathbf{x})$.

The real vector space of all sk-splines on $\Lambda_{\mathbf{n}}$, associated with the kernel K will be denoted by $SK(\Lambda_{\mathbf{n}})$. As the vector space V generated by the set of functions $\{1, K(\mathbf{x}-\mathbf{x_k}),\ \mathbf{k} \in \Omega_{\mathbf{n}}\}$ has dimension at most $N+1$ and $SK(\Lambda_{\mathbf{n}})$ is the subspace of V formed by the functions whose coefficients satisfy the condition $\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{k}} = 0$, then $\dim SK(\Lambda_{\mathbf{n}}) \le N$.

The next four lemmas will not be proved, because they are of simple verification.

Lemma 3.1. *Let* $\mathbf{l} \in \mathbb{Z}^d$. *Then*

$$\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} e^{i\mathbf{l}\cdot\mathbf{x_k}} = \begin{cases} N, & \mathbf{l} \equiv \mathbf{0} \bmod(2\mathbf{n}), \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \int_{\mathbb{T}^d} e^{i\mathbf{l}\cdot\mathbf{x}} d\nu(\mathbf{x}) = \begin{cases} 0, & \mathbf{l} \neq \mathbf{0}, \\ 1, & \mathbf{l} = \mathbf{0}. \end{cases}$$

Lemma 3.2. *For every* $\mathbf{l} \in \mathbb{Z}^d$,

$$\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} \cos(\mathbf{l}\cdot\mathbf{x_k}) = \begin{cases} N, & \mathbf{l} \equiv \mathbf{0} \bmod(2\mathbf{n}), \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} \sin(\mathbf{l}\cdot\mathbf{x_k}) = 0.$$

Lemma 3.3. *For every* $\mathbf{l}, \mathbf{j} \in \mathbb{Z}^d$ *we have that*

$$\sum_{\mathbf{k}\in\Omega_\mathbf{n}} (\cos(\mathbf{j}\cdot\mathbf{x_k}))(\cos(\mathbf{l}\cdot\mathbf{x_k})) = \begin{cases} N, & \mathbf{l}+\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n}) \text{ and} \\ & \mathbf{l}-\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n}), \\ N/2, & \mathbf{l}+\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n}) \text{ or} \\ & \mathbf{l}-\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n}), \\ 0, & \text{otherwise,} \end{cases}$$

$$\sum_{\mathbf{k}\in\Omega_\mathbf{n}} (\sin(\mathbf{j}\cdot\mathbf{x_k}))(\sin(\mathbf{l}\cdot\mathbf{x_k})) = \begin{cases} N/2, & \mathbf{l}+\mathbf{j} \not\equiv \mathbf{0} \bmod(2\mathbf{n}) \text{ and} \\ & \mathbf{l}-\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n}), \\ -N/2, & \mathbf{l}+\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n}) \text{ and} \\ & \mathbf{l}-\mathbf{j} \not\equiv \mathbf{0} \bmod(2\mathbf{n}), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sum_{\mathbf{k}\in\Omega_\mathbf{n}} (\cos(\mathbf{j}\cdot\mathbf{x_k}))(\sin(\mathbf{l}\cdot\mathbf{x_k})) = 0.$$

Definition 3.4. *For* $K \in C(\mathbb{T}^d)$, $\mathbf{j} \in \mathbb{Z}^d$ *and* $\mathbf{x} \in \mathbb{T}^d$, *we define*

$$\lambda_\mathbf{j}(\mathbf{x}) = \sum_{\mathbf{k}\in\Omega_\mathbf{n}} e^{i\mathbf{j}\cdot\mathbf{x_k}} K(\mathbf{x}-\mathbf{x_k}),$$

$$\rho_\mathbf{j}(\mathbf{x}) = \frac{2}{N}\mathrm{Re}(\lambda_\mathbf{j}(\mathbf{x})) = \frac{2}{N}\sum_{\mathbf{k}\in\Omega_\mathbf{n}} (\cos(\mathbf{j}\cdot\mathbf{x_k}))K(\mathbf{x}-\mathbf{x_k})$$

and

$$\sigma_\mathbf{j}(\mathbf{x}) = \frac{2}{N}\mathrm{Im}(\lambda_\mathbf{j}(\mathbf{x})) = \frac{2}{N}\sum_{\mathbf{k}\in\Omega_\mathbf{n}} (\sin(\mathbf{j}\cdot\mathbf{x_k}))K(\mathbf{x}-\mathbf{x_k}).$$

Lemma 3.5. *Let* $\mathbf{p}, \mathbf{j} \in \mathbb{Z}^d$. *Then for every* $\mathbf{x} \in \mathbb{T}^d$,

$$\rho_{2\mathbf{np}+\mathbf{j}}(\mathbf{x}) = \rho_\mathbf{j}(\mathbf{x}), \quad \rho_{2\mathbf{np}-\mathbf{j}}(\mathbf{x}) = \rho_\mathbf{j}(\mathbf{x}), \quad \rho_{-\mathbf{j}}(\mathbf{x}) = \rho_\mathbf{j}(\mathbf{x}),$$

$$\sigma_{2\mathbf{np}+\mathbf{j}}(\mathbf{x}) = \sigma_\mathbf{j}(\mathbf{x}), \quad \sigma_{2\mathbf{np}-\mathbf{j}}(\mathbf{x}) = -\sigma_\mathbf{j}(\mathbf{x}), \quad \sigma_{-\mathbf{j}}(\mathbf{x}) = -\sigma_\mathbf{j}(\mathbf{x}).$$

Theorem 3.6. *Consider a kernel K given by $K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}}$, where $(a_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$ is a sequence of real numbers such that $\sum_{\mathbf{l}\in\mathbb{Z}^d} |a_{\mathbf{l}}| < \infty$ and $a_{\mathbf{l}} = a_{-\mathbf{l}}$ for every $\mathbf{l} \in \mathbb{Z}^d$. Then K is a real function, continuous, even and for every $\mathbf{j} \in \mathbb{Z}^d$,*

$$\rho_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}} \cos((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) + a_{2\mathbf{np}-\mathbf{j}} \cos((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x})),$$

$$\sigma_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}} \sin((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) - a_{2\mathbf{np}-\mathbf{j}} \sin((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x})).$$

Proof: For each $m \in \mathbb{N}$, let $K_m(\mathbf{x}) = \sum_{|\mathbf{l}|_2\le m} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}}$. We have that $(K_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $C(\mathbb{T}^d)$ and as $C(\mathbb{T}^d)$ is complete, there is a function $K \in C(\mathbb{T}^d)$ such that $K_m \to K$ uniformly. We have that

$$K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} \cos(\mathbf{l}\cdot\mathbf{x}) \tag{4}$$

and K is a real and even function. Fix $\mathbf{j} \in \mathbb{Z}^d$ and let

$$A_{\mathbf{j}} = \{\mathbf{l} \in \mathbb{Z}^d : \mathbf{l}+\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n})\} = \{2\mathbf{np}-\mathbf{j} : \mathbf{p} \in \mathbb{Z}^d\},$$

$$B_{\mathbf{j}} = \{\mathbf{l} \in \mathbb{Z}^d : \mathbf{l}-\mathbf{j} \equiv \mathbf{0} \bmod(2\mathbf{n})\} = \{2\mathbf{np}+\mathbf{j} : \mathbf{p} \in \mathbb{Z}^d\}.$$

From Lemma 3.3 we have that

$$\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{j}\cdot\mathbf{x_k}))(\cos(\mathbf{l}\cdot\mathbf{x_k})) = \begin{cases} N, & \mathbf{l} \in A_{\mathbf{j}} \cap B_{\mathbf{j}}, \\ N/2, & \mathbf{l} \in A_{\mathbf{j}} \triangle B_{\mathbf{j}}, \\ 0, & \mathbf{l} \in (A_{\mathbf{j}} \cup B_{\mathbf{j}})^c \end{cases} \tag{5}$$

$$\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\sin(\mathbf{j}\cdot\mathbf{x_k}))(\sin(\mathbf{l}\cdot\mathbf{x_k})) = \begin{cases} N/2, & \mathbf{l} \in B_{\mathbf{j}} \backslash A_{\mathbf{j}}, \\ -N/2, & \mathbf{l} \in A_{\mathbf{j}} \backslash B_{\mathbf{j}}, \\ 0, & \mathbf{l} \in (A_{\mathbf{j}} \triangle B_{\mathbf{j}})^c. \end{cases} \tag{6}$$

Now using (4) we obtain

$$\begin{aligned} \lambda_{\mathbf{j}}(\mathbf{x}) &= \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} e^{i\mathbf{j}\cdot\mathbf{x}_{\mathbf{k}}} K(\mathbf{x}-\mathbf{x}_{\mathbf{k}}) \\ &= \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{j}\cdot\mathbf{x}_{\mathbf{k}}))(\cos(\mathbf{l}\cdot(\mathbf{x}-\mathbf{x}_{\mathbf{k}}))) \\ &+ i\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\sin(\mathbf{j}\cdot\mathbf{x}_{\mathbf{k}}))(\cos(\mathbf{l}\cdot(\mathbf{x}-\mathbf{x}_{\mathbf{k}}))). \end{aligned} \tag{7}$$

Thus by (7), from Lemma 3.3 and from (5)

$$\begin{aligned} \rho_{\mathbf{j}}(\mathbf{x}) &= \frac{2}{N}\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{j}\cdot\mathbf{x}_{\mathbf{k}}))(\cos(\mathbf{l}\cdot\mathbf{x}-\mathbf{l}\cdot\mathbf{x}_{\mathbf{k}})) \\ &= \frac{2}{N}\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}}(\cos(\mathbf{l}\cdot\mathbf{x})) \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{j}\cdot\mathbf{x}_{\mathbf{k}}))(\cos(\mathbf{l}\cdot\mathbf{x}_{\mathbf{k}})) \\ &= \frac{2}{N}\sum_{\mathbf{l}\in A_{\mathbf{j}}\cap B_{\mathbf{j}}} N a_{\mathbf{l}}\cos(\mathbf{l}\cdot\mathbf{x}) + \frac{2}{N}\sum_{\mathbf{l}\in A_{\mathbf{j}}\triangle B_{\mathbf{j}}} \frac{N}{2} a_{\mathbf{l}}\cos(\mathbf{l}\cdot\mathbf{x}) \\ &= \sum_{2\mathbf{np}+\mathbf{j}\in A_{\mathbf{j}}\cap B_{\mathbf{j}},\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}+\mathbf{j}}\cos((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) \\ &+ \sum_{2\mathbf{np}-\mathbf{j}\in A_{\mathbf{j}}\cap B_{\mathbf{j}},\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{j}}\cos((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x}) \\ &+ \sum_{2\mathbf{np}+\mathbf{j}\in A_{\mathbf{j}}\triangle B_{\mathbf{j}},\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}+\mathbf{j}}\cos((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) \\ &+ \sum_{2\mathbf{np}-\mathbf{j}\in A_{\mathbf{j}}\triangle B_{\mathbf{j}},\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{j}}\cos((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x}) \\ &= \sum_{2\mathbf{np}+\mathbf{j}\in A_{\mathbf{j}}\cup B_{\mathbf{j}},\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}+\mathbf{j}}\cos((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) \\ &+ \sum_{2\mathbf{np}-\mathbf{j}\in A_{\mathbf{j}}\cup B_{\mathbf{j}},\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{j}}\cos((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x}) \\ &= \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}+\mathbf{j}}\cos((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) + \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{j}}\cos((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x}). \end{aligned}$$

In an analogous way, using (7), Lemma 3.3 and (6) we obtain

$$\sigma_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}+\mathbf{j}}\sin((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x}) - \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{j}}\sin((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x}),$$

and this concludes the proof. $\square$

4 Fundamental sk-spline

Definition 4.1. *Suppose that $\rho_{\mathbf{j}}(\mathbf{0}) \neq 0$ for all $\mathbf{j} \in \Omega_{\mathbf{n}}$, $\mathbf{j} \neq \mathbf{0}$. We define $\widetilde{sk}_{\mathbf{n}}$ by*

$$\widetilde{sk}_{\mathbf{n}}(\mathbf{x}) = \frac{1}{N} + \frac{1}{N} \sum_{\mathbf{j} \in \Omega_{\mathbf{n}}^*} \frac{\rho_{\mathbf{j}}(\mathbf{x})}{\rho_{\mathbf{j}}(\mathbf{0})}$$

where $\Omega_{\mathbf{n}}^* = \Omega_{\mathbf{n}} \setminus \{(0, \ldots, 0)\}$.

Lemma 4.2. *The function $\widetilde{sk}_{\mathbf{n}}$ is a sk-spline.*

Proof: We have, by the definition of $\rho_{\mathbf{j}}(\mathbf{x})$ that

$$\begin{aligned} \widetilde{sk}_{\mathbf{n}}(\mathbf{x}) &= \frac{1}{N} + \frac{1}{N} \sum_{\mathbf{j} \in \Omega_{\mathbf{n}}^*} \frac{\rho_{\mathbf{j}}(\mathbf{x})}{\rho_{\mathbf{j}}(\mathbf{0})} \\ &= \frac{1}{N} + \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} \left(\frac{2}{N^2} \sum_{\mathbf{j} \in \Omega_{\mathbf{n}}^*} \frac{1}{\rho_{\mathbf{j}}(\mathbf{0})} (\cos(\mathbf{j} \cdot \mathbf{x}_{\mathbf{k}})) \right) K(\mathbf{x} - \mathbf{x}_{\mathbf{k}}) \\ &= \frac{1}{N} + \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} c_{\mathbf{k}} K(\mathbf{x} - \mathbf{x}_{\mathbf{k}}) \end{aligned}$$

and by Lemma 3.2

$$\sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} c_{\mathbf{k}} = \frac{2}{N^2} \sum_{\mathbf{j} \in \Omega_{\mathbf{n}}^*} \frac{1}{\rho_{\mathbf{j}}(\mathbf{0})} \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} \cos(\mathbf{j} \cdot \mathbf{x}_{\mathbf{k}}) = 0.$$

Thus $\widetilde{sk}_{\mathbf{n}}$ is a $sk-$spline by definition. $\square$

The sk-spline $\widetilde{sk}_{\mathbf{n}}$ will be called fundamental sk-spline.

Lemma 4.3. *If $\rho_{\mathbf{j}}(\mathbf{0}) \neq 0$ for all $\mathbf{j} \in \Omega_{\mathbf{n}}^*$, then the sk-spline $\widetilde{sk}_{\mathbf{n}}$ satisfy*

$$\widetilde{sk}_{\mathbf{n}}(\mathbf{x}_{\mathbf{k}}) = \begin{cases} 1, & \mathbf{k} = \mathbf{0}, \\ 0, & \mathbf{k} \in \Omega_{\mathbf{n}}^*. \end{cases}$$

Proof: From Theorem 3.6 we have that

$$\begin{aligned}
\rho_{\mathbf{j}}(\mathbf{x_l}) &= \sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}}\cos((2\mathbf{np}+\mathbf{j})\cdot\mathbf{x_l}) + a_{2\mathbf{np}-\mathbf{j}}\cos((2\mathbf{np}-\mathbf{j})\cdot\mathbf{x_l})) \\
&= \sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}}\cos(\mathbf{j}\cdot\mathbf{x_l}) + a_{2\mathbf{np}-\mathbf{j}}\cos(\mathbf{j}\cdot\mathbf{x_l})) \\
&= (\cos(\mathbf{j}\cdot\mathbf{x_l}))\sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}} + a_{2\mathbf{np}-\mathbf{j}}) \\
&= (\cos(\mathbf{j}\cdot\mathbf{x_l}))\rho_{\mathbf{j}}(\mathbf{0}),
\end{aligned}$$

then by Lemma 3.2

$$\widetilde{sk}_{\mathbf{n}}(\mathbf{x_k}) = \frac{1}{N} + \frac{1}{N}\sum_{\mathbf{j}\in\Omega_{\mathbf{n}}^*} \frac{(\cos(\mathbf{j}\cdot\mathbf{x_k}))\rho_{\mathbf{j}}(\mathbf{0})}{\rho_{\mathbf{j}}(\mathbf{0})} = \frac{1}{N}\sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} \cos(\mathbf{k}\cdot\mathbf{x_j}) = \begin{cases} 1, & \mathbf{k}=\mathbf{0}, \\ 0, & \mathbf{k}\in\Omega_{\mathbf{n}}^*, \end{cases}$$

and then we proved the lemma. □

Definition 4.4. *Let f be a function defined on $\mathbb{T}^d$ and let $\{\mathbf{y_j} : \mathbf{j}\in\Omega_{\mathbf{n}}\}\subset\mathbb{T}^d$. If there are constants $c^*, c^*_{\mathbf{k}}\in\mathbb{R}$, such that*

$$sk_{\mathbf{n}}(f,\mathbf{y_j}) = c^* + \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c^*_{\mathbf{k}} K(\mathbf{y_j}-\mathbf{x_k}) = f(\mathbf{y_j}), \quad \mathbf{j}\in\Omega_{\mathbf{n}},$$

we say that the sk-spline

$$sk_{\mathbf{n}}(f,\mathbf{x}) = c^* + \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c^*_{\mathbf{k}} K(\mathbf{x}-\mathbf{x_k})$$

is an interpolating sk-spline of f with knots $\mathbf{x_k}$ and interpolation points $\mathbf{y_k}$.

Theorem 4.5. *Suppose $\rho_{\mathbf{j}}(\mathbf{0})\neq 0$ for any $\mathbf{j}\in\Omega_{\mathbf{n}}^*$. Then for any function f defined on $\mathbb{T}^d$, there is an unique interpolating sk-spline of f with knots and interpolation points $\mathbf{x_k}, \mathbf{k}\in\Omega_{\mathbf{n}}$, that can be written in the form*

$$sk_{\mathbf{n}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} f(\mathbf{x_k})\widetilde{sk}_{\mathbf{n}}(\mathbf{x}-\mathbf{x_k}).$$

Proof: Let $c_{\mathbf{k}},\ \mathbf{k} \in \Omega_{\mathbf{n}}$ be the coefficients of the sk-spline $\widetilde{sk}_{\mathbf{n}}$ that were obtained in the proof of Lemma 4.2 and let

$$sk_{\mathbf{n}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} \frac{f(\mathbf{x}_{\mathbf{k}})}{N} + \sum_{\mathbf{l}\in\Omega_{\mathbf{n}}} \left(\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{l}} f(\mathbf{x}_{\mathbf{k}}) \right) K(\mathbf{x}-\mathbf{x}_{\mathbf{l}}) = d + \sum_{\mathbf{l}\in\Omega_{\mathbf{n}}} d_{\mathbf{l}} K(\mathbf{x}-\mathbf{x}_{\mathbf{l}}).$$

Since $\sum_{\mathbf{l}\in\Omega_{\mathbf{n}}} c_{\mathbf{l}} = 0$, it follows that $\sum_{\mathbf{l}\in\Omega_{\mathbf{n}}} d_{\mathbf{l}} = 0$, and thus $sk_{\mathbf{n}}(f,\cdot)$ is a sk-spline. Applying Lemma 4.3 we obtain that

$$sk_{\mathbf{n}}(f,\mathbf{x}_{\mathbf{l}}) = \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} f(\mathbf{x}_{\mathbf{k}}) \widetilde{sk}_{\mathbf{n}}(\mathbf{x}_{\mathbf{l}}-\mathbf{x}_{\mathbf{k}}) = f(\mathbf{x}_{\mathbf{l}})$$

for any $\mathbf{l} \in \Omega_{\mathbf{n}}$. Then we can conclude that $sk_{\mathbf{n}}(f,\cdot)$ is an interpolating sk-spline of f with knots and interpolation points $\mathbf{x}_{\mathbf{k}}$.

Let $\{\mathbf{w}_j : 1 \le j \le N\}$ be an enumeration of $\Lambda_{\mathbf{n}}$. Then for every function f on $\mathbb{T}^d$, there are constants $c_1, c_2, \ldots, c_{N+1} \in \mathbb{R}$ satisfying $\sum_{l=1}^{N} c_l = 0$, such that

$$sk_{\mathbf{n}}(f,\mathbf{x}) = c_{N+1} + \sum_{l=1}^{N} c_l K(\mathbf{x}-\mathbf{w}_l)$$

is an interpolating sk-spline of f. Given $y_1, y_2, \ldots, y_N \in \mathbb{R}$, let

$$\alpha_j = \left(\prod_{1\le l\le N,\ l\ne j} |\mathbf{w}_j - \mathbf{w}_l|_2 \right)^{-1},\ 1 \le j \le N,$$

and $g : \mathbb{T}^d \to \mathbb{R}$ defined by

$$g(\mathbf{x}) = \sum_{j=1}^{N} y_j \alpha_j \prod_{1\le l\le N\ l\ne j} |\mathbf{x}-\mathbf{w}_l|_2, \quad \mathbf{x} \in \mathbb{T}^d.$$

Thus $g(\mathbf{w}_k) = y_k$, for $1 \le k \le N$. Then there are $c_1, \ldots, c_N, c_{N+1} \in \mathbb{R}$ such that the $sk-$spline

$$sk_{\mathbf{n}}(\mathbf{x}) = c_{N+1} + \sum_{l=1}^{N} c_l K(\mathbf{x}-\mathbf{w}_l)$$

is an interpolating $sk-$spline of g, that is,

$$c_{N+1}+\sum_{l=1}^{N}c_lK(\mathbf{w}_k-\mathbf{w}_l)=g(\mathbf{w}_k)=y_k,\quad 1\le k\le N.$$

Let

$$\widetilde{\mathbf{K}}=\begin{pmatrix} K(\mathbf{w}_1-\mathbf{w}_1) & \cdots & K(\mathbf{w}_1-\mathbf{w}_N) & 1\\ \vdots & \vdots & \vdots & \vdots\\ K(\mathbf{w}_N-\mathbf{w}_1) & \cdots & K(\mathbf{w}_N-\mathbf{w}_N) & 1\\ 1 & \cdots & 1 & 0\end{pmatrix},$$

$\mathbf{C}=(c_1,c_2,\ldots,c_{N+1})$, $\mathbf{Y}=(y_1,y_2,\ldots,y_{N+1})\in\mathbb{R}^{N+1}$ and

$$\widetilde{\mathbf{K}}=\begin{pmatrix}\mathbf{K} & \mathbf{u}\\ \mathbf{u}^t & 0\end{pmatrix},\quad \mathbf{u}=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},\quad \mathbf{C}^t=\begin{pmatrix}c_1\\ \vdots\\ c_{N+1}\end{pmatrix},\quad \mathbf{Y}^t=\begin{pmatrix}y_1\\ \vdots\\ y_{N+1}\end{pmatrix},$$

where $\mathbf{u}$ is a $N\times 1$ matrix. Let $W=\left\{\mathbf{C}=(c_1,c_2,\ldots,c_{N+1})\in\mathbb{R}^{N+1}:\sum_{l=1}^{N}c_l=0\right\}$. Then for every $\mathbf{Y}\in\mathbb{R}^{N+1}$ with $y_{N+1}=0$, there is $\mathbf{C}\in W$ such that $\widetilde{\mathbf{K}}\mathbf{C}^t=\mathbf{Y}^t$.

Now we consider the linear map $T:W\to\mathbb{R}^{N+1}$ defined by $T(\mathbf{C})=(\widetilde{\mathbf{K}}\mathbf{C}^t)^t$. We can conclude that T is injective, since T is linear and $\dim W=N=\dim \mathrm{Im(T)}$.

Let f be a function on $\mathbb{T}^d$ and suppose that there are $C=(c_1,c_2,\ldots,c_{N+1})$, $\overline{C}=(\bar{c}_1,\bar{c}_2,\ldots,\bar{c}_{N+1})\in W$ such that

$$sk_{\mathbf{n}}(f,\mathbf{x})=c_{N+1}+\sum_{l=1}^{N}c_lK(\mathbf{x}-\mathbf{w}_l)\quad and\quad \overline{sk}_{\mathbf{n}}(f,\mathbf{x})=\bar{c}_{N+1}+\sum_{l=1}^{N}\bar{\mathbf{c}}_lK(\mathbf{x}-\mathbf{w}_l)$$

are two interpolating sk-splines of f. If $F=(f(\mathbf{w}_1),\ldots,f(\mathbf{w}_N),0)$, then we have $T(C)=(\widetilde{\mathbf{K}}\mathbf{C}^t)^t=F$ and $T(\overline{C})=(\widetilde{\mathbf{K}}\overline{\mathbf{C}}^t)^t=F$. Since T is injective, it follows that $C=\overline{C}$, that is, $sk_{\mathbf{n}}(f,\mathbf{x})=\overline{sk}_{\mathbf{n}}(f,\mathbf{x})$, for all $\mathbf{x}\in\mathbb{T}^d$. □

Remark 4.6. *Let K be a kernel satisfying the conditions of the Theorem 3.6 and $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$. Suppose $\rho_{\mathbf{j}}(0) \neq 0$ for all $\mathbf{j} \in \Omega_{\mathbf{n}}$. Then the vector space $SK(\Lambda_{\mathbf{n}})$ of all sk-splines on $\Lambda_{\mathbf{n}}$ and associated with the kernel K has dimension $N = 2^d n_1 n_2 \ldots n_d$. In particular if $n_1 = n_2 = \cdots = n_d = n$ we have* $\dim(SK(\Lambda_{\mathbf{n}})) = (2n)^d$.

5 Approximation by sk-splines

In this section we will prove the main result of this paper, the Theorem 5.7. This theorem says how a function of the type $f = K * \phi$, for $\phi \in L^p(\mathbb{T}^d)$, can be approximated by the sk-splines $sk_{\mathbf{n}}(f, \cdot)$ in the space $L^q(\mathbb{T}^d)$, where $1 \leq p \leq 2 \leq q \leq \infty$ with $1/p - 1/q \geq 1/2$. But for our applications, the most interesting result is the Corollary 5.9, since its hypothesis can be easily verified.

In all results of this section, we consider a kernel K as in the Theorem 3.6 and such that $\rho_{\mathbf{j}}(0) \neq 0$ for all $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{j} \in \Omega_{\mathbf{n}}$.

The following result can be easily verified.

Lemma 5.1. *For $\mathbf{j} \in \Omega_{\mathbf{n}}$ and $\mathbf{l} \in \mathbb{Z}^d$ we have that $\rho_{\mathbf{l}}(\mathbf{x} - \mathbf{x_j}) = \rho_{\mathbf{l}}(\mathbf{x}) \cos(\mathbf{l} \cdot \mathbf{x_j}) + \sigma_{\mathbf{l}}(\mathbf{x}) \sin(\mathbf{l} \cdot \mathbf{x_j})$.*

Lemma 5.2. *For every $\mathbf{l} \in \mathbb{Z}^d$, $\mathbf{l} \not\equiv \mathbf{0} \bmod(2\mathbf{n})$ and $\mathbf{x} \in \mathbb{T}^d$,*

$$\sum_{\mathbf{j} \in \Omega_{\mathbf{n}}} e^{i\mathbf{l}\cdot\mathbf{x_j}} \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_j}) = \frac{\lambda_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})}.$$

Proof: Firstly we will prove the result for the real part. Consider the sets $A_{\mathbf{l}}$ and $B_{\mathbf{l}}$ introduced in the proof of Theorem 3.6.

Using Lemma 5.1 together with Lemmas 3.2 and 3.3, the equation (5) and the fact that $\mathbf{l} \not\equiv \mathbf{0} \bmod (2\mathbf{n})$, we have that

$$\begin{aligned}
\sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j}))\widetilde{sk}_{\mathbf{n}}(\mathbf{x}-\mathbf{x_j}) &= \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j})) \left\{ \frac{1}{N} + \frac{1}{N} \sum_{\mathbf{k}\in\Omega^*_{\mathbf{n}}} \frac{\rho_{\mathbf{k}}(\mathbf{x}-\mathbf{x_j})}{\rho_{\mathbf{k}}(\mathbf{0})} \right\} \\
&= \frac{1}{N} \sum_{\mathbf{k}\in\Omega^*_{\mathbf{n}}} \frac{\rho_{\mathbf{k}}(\mathbf{x})}{\rho_{\mathbf{k}}(\mathbf{0})} \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j}))(\cos(\mathbf{k}\cdot\mathbf{x_j})) \\
&= \frac{1}{N} \sum_{\mathbf{k}\in\Omega^*_{\mathbf{n}}\cap(A_{\mathbf{l}}\cap B_{\mathbf{l}})} \frac{\rho_{\mathbf{k}}(\mathbf{x})}{\rho_{\mathbf{k}}(\mathbf{0})} \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j}))(\cos(\mathbf{k}\cdot\mathbf{x_j})) \\
&+ \frac{1}{N} \sum_{\mathbf{k}\in\Omega^*_{\mathbf{n}}\cap(A_{\mathbf{l}}\Delta B_{\mathbf{l}})} \frac{\rho_{\mathbf{k}}(\mathbf{x})}{\rho_{\mathbf{k}}(\mathbf{0})} \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j}))(\cos(\mathbf{k}\cdot\mathbf{x_j})) \\
&= \sum_{\mathbf{k}\in\Omega^*_{\mathbf{n}}\cap(A_{\mathbf{l}}\cap B_{\mathbf{l}})} \frac{\rho_{\mathbf{k}}(\mathbf{x})}{\rho_{\mathbf{k}}(\mathbf{0})} + \frac{1}{2} \sum_{\mathbf{k}\in\Omega^*_{\mathbf{n}}\cap(A_{\mathbf{l}}\Delta B_{\mathbf{l}})} \frac{\rho_{\mathbf{k}}(\mathbf{x})}{\rho_{\mathbf{k}}(\mathbf{0})}. \qquad (8)
\end{aligned}$$

If $\mathbf{k} \in B_{\mathbf{l}}$ then there is $\mathbf{p} \in \mathbb{Z}^d$ such that $\mathbf{k} = 2\mathbf{np} + \mathbf{l}$ and thus $\rho_{\mathbf{k}}(\mathbf{x}) = \rho_{2\mathbf{np}+\mathbf{l}}(\mathbf{x}) = \rho_{\mathbf{l}}(\mathbf{x})$ by Lemma 3.5. In an analogous way, if $\mathbf{k} \in A_{\mathbf{l}}$ we can conclude that $\rho_{\mathbf{k}}(\mathbf{x}) = \rho_{\mathbf{l}}(\mathbf{x})$. Then, by (8) we have that

$$\sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j}))\widetilde{sk}_{\mathbf{n}}(\mathbf{x}-\mathbf{x_j}) = \frac{\rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})} \left(\#(\Omega^*_{\mathbf{n}} \cap (A_{\mathbf{l}} \cap B_{\mathbf{l}})) + \frac{1}{2}\#(\Omega^*_{\mathbf{n}} \cap (A_{\mathbf{l}}\Delta B_{\mathbf{l}})) \right). \qquad (9)$$

Let $\mathbf{l} = (l_1, \ldots, l_d) \in \mathbb{Z}^d$. For each $1 \leq j \leq d$, there is an unique q_j and an unique r_j satisfying $q_j, r_j \in \mathbb{Z}, 0 \leq r_j \leq 2n_j - 1$ and $l_j = 2n_j q_j + r_j$. Then $\mathbf{l} = 2\mathbf{nq} + \mathbf{r}$ where $\mathbf{q} = (q_1, \ldots, q_d) \in \mathbb{Z}^d$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \Omega_{\mathbf{n}}$, so

$$B_{\mathbf{l}} = \{2\mathbf{np}+\mathbf{l} : \mathbf{p} \in \mathbb{Z}^d\} = \{2\mathbf{n}(\mathbf{p}+\mathbf{q})+\mathbf{r} : \mathbf{p} \in \mathbb{Z}^d\} = \{2\mathbf{np}+\mathbf{r} : \mathbf{p} \in \mathbb{Z}^d\} = B_{\mathbf{r}}.$$

In an analogous way we have $A_{\mathbf{l}} = \{2\mathbf{np} - \mathbf{r} : \mathbf{p} \in \mathbb{Z}^d\} = A_{\mathbf{r}}$. As $\rho_{\mathbf{l}}(\mathbf{x}) = \rho_{\mathbf{r}}(\mathbf{x})$ by Lemma 3.5, we obtain by (9) that

$$\sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{l}\cdot\mathbf{x_j}))\widetilde{sk}_{\mathbf{n}}(\mathbf{x}-\mathbf{x_j}) = \frac{\rho_{\mathbf{r}}(\mathbf{x})}{\rho_{\mathbf{r}}(\mathbf{0})} \left(\#\left(\Omega^*_{\mathbf{n}} \cap (A_{\mathbf{r}} \cap B_{\mathbf{r}})\right) + \frac{1}{2}\#(\Omega^*_{\mathbf{n}} \cap (A_{\mathbf{r}}\Delta B_{\mathbf{r}})) \right).$$

Then it is enough to prove the result for $\mathbf{l} \in \Omega^*_{\mathbf{n}}$.

Let $\mathbf{l} = (l_1, \dots, l_d), \mathbf{k} = (k_1, \dots, k_d) \in \Omega^*_{\mathbf{n}}$. Then $\mathbf{l} - \mathbf{k} \equiv \mathbf{0} \mod (2\mathbf{n})$ if and only if $\mathbf{k} = \mathbf{l}$, and $\mathbf{l} + \mathbf{k} \equiv \mathbf{0} \mod (2\mathbf{n})$ if and only if $k_j = l_j = 0$ and $k_j = 2n_j - l_j$ if $l_j \neq 0$. Then $\mathbf{k} \in \Omega^*_{\mathbf{n}} \cap (A_{\mathbf{l}} \cap B_{\mathbf{l}})$ if and only if $\mathbf{k} = \mathbf{l}$ and $l_j \in \{0, n_j\}$ for all $1 \leq j \leq d$; $\mathbf{k} \in \Omega^*_{\mathbf{n}} \cap (B_{\mathbf{l}} \setminus A_{\mathbf{l}})$ if and only if $\mathbf{k} = \mathbf{l}$ and $l_j \notin \{0, n_j\}$ for some $1 \leq j \leq d$; $\mathbf{k} \in \Omega^*_{\mathbf{n}} \cap (A_{\mathbf{l}} \setminus B_{\mathbf{l}})$ if and only if $k_j = l_j = 0,\ k_j = 2n_j - l_j$ if $l_j \neq 0$ and $l_j \notin \{0, n_j\}$ for some $1 \leq j \leq d$. Let

$$\mathcal{A} = \{\mathbf{l} \in \Omega^*_{\mathbf{n}} : l_j \in \{0, n_j\} \text{ for all } 1 \leq j \leq d\},$$

$$\mathcal{B} = \{\mathbf{l} \in \Omega^*_{\mathbf{n}} : l_j \notin \{0, n_j\} \text{ for all } 1 \leq j \leq d\}.$$

Then $\Omega^*_{\mathbf{n}} = \mathcal{A} \cup \mathcal{B},\ \mathcal{A} \cap \mathcal{B} = \varnothing$ and

$$\#(\Omega^*_{\mathbf{n}} \cap (A_{\mathbf{l}} \cap B_{\mathbf{l}})) = \begin{cases} 1, & \mathbf{l} \in \mathcal{A}, \\ 0, & \mathbf{l} \in \mathcal{B}, \end{cases}, \qquad \#(\Omega^*_{\mathbf{n}} \cap (A_{\mathbf{l}} \Delta B_{\mathbf{l}})) = \begin{cases} 0, & \mathbf{l} \in \mathcal{A}, \\ 2, & \mathbf{l} \in \mathcal{B}. \end{cases}$$

Then it follow from (9) that for all $\mathbf{l} \in \Omega^*_{\mathbf{n}}$ we have

$$\sum_{\mathbf{j} \in \Omega_{\mathbf{n}}} (\cos(\mathbf{l} \cdot \mathbf{x_j})) \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_j}) = \frac{\rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})}. \tag{10}$$

In an analogous way, for the imaginary part, we obtain that for all $\mathbf{l} \in \mathbb{Z}^d$, $\mathbf{l} \neq \mathbf{0}$,

$$\sum_{\mathbf{j} \in \Omega_{\mathbf{n}}} (\sin(\mathbf{l} \cdot \mathbf{x_j})) \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_j}) = \frac{\sigma_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})},$$

and this concludes the proof. □

Remark 5.3. *Let $|\cdot|$ be a norm on $\mathbb{R}^d$ and let $K \in C(\mathbb{T}^d)$ be a kernel as in Theorem 3.6, such that $a_{\mathbf{l}} = a_{\mathbf{k}}$ if $\mathbf{l}, \mathbf{k} \in \mathbb{Z}^d$ and $|\mathbf{l}| = |\mathbf{k}|$. Given $\mathbf{k} = (k_1, \dots, k_d)$, $\mathbf{p} = (p_1, \dots, p_d)$, $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{Z}^d$, let $\overline{\mathbf{k}} = ((-1)^{i_1} k_1, \dots, (-1)^{i_d} k_d)$ and $\overline{\mathbf{p}} = ((-1)^{i_1} p_1, \dots, (-1)^{i_d} p_d)$. Then $|2\mathbf{n}\mathbf{p} + \overline{\mathbf{k}}| = |2\mathbf{n}\overline{\mathbf{p}} + \mathbf{k}|$ and so $a_{2\mathbf{n}\mathbf{p}+\overline{\mathbf{k}}} = a_{2\mathbf{n}\overline{\mathbf{p}}+\mathbf{k}}$.*

Given $\mathbf{j} = (j_1, \dots, j_d) \in (\mathbb{N} \cup \{0\})^d = \{0, 1, 2, \dots\}^d$, let

$$D_{\mathbf{j}} = \left\{ \mathbf{p} = (p_1, \dots, p_d) \in \mathbb{Z}^d : |p_i| = j_i,\ i = 1, 2, \dots, d \right\}.$$

Then

$$\begin{aligned}\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\mathbf{p}+\bar{\mathbf{k}}} &= \sum_{\mathbf{j}\in(\mathbb{N}\cup\{0\})^d}\sum_{\mathbf{p}\in D_{\mathbf{j}}} a_{2\mathbf{n}\mathbf{p}+\bar{\mathbf{k}}} = \sum_{\mathbf{j}\in(\mathbb{N}\cup\{0\})^d}\sum_{\mathbf{p}\in D_{\mathbf{j}}} a_{2\mathbf{n}\bar{\mathbf{p}}+\mathbf{k}} \\ &= \sum_{\mathbf{j}\in(\mathbb{N}\cup\{0\})^d}\sum_{\mathbf{p}\in D_{\mathbf{j}}} a_{2\mathbf{n}\mathbf{p}+\mathbf{k}} = \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\mathbf{p}+\mathbf{k}}.\end{aligned}$$

Remark 5.4. *Consider a kernel K given by $K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}}$, such that $a_{\mathbf{l}} \geq 0$, for all $\mathbf{l} \in \mathbb{Z}^d$ and $\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\mathbf{p}-\mathbf{k}} \leq C a_{2\mathbf{n}-\mathbf{k}}$, for all $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ and every $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, with $0 \leq k_j \leq n_j$, for $j = 1, 2, \ldots, d$, where C is a positive constant independent of $\mathbf{n}$ and $\mathbf{k}$. Then $\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} < \infty$. If $a_{\mathbf{l}} = a_{-\mathbf{l}}$ for all $\mathbf{l} \in \mathbb{Z}^d$, then by Theorem 3.6, the kernel K is a real, continuous and even function.*

Lemma 5.5. *Let $|\cdot|$ be a norm on $\mathbb{R}^d$ and let K be the kernel given by $K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}}$, where $(a_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$ is a sequence with $a_{\mathbf{l}} = a_{\mathbf{k}}$ if $|\mathbf{l}| = |\mathbf{k}|$ and $a_{\mathbf{l}} \geq a_{\mathbf{k}} > 0$ if $|\mathbf{l}| \geq |\mathbf{k}|$, for $\mathbf{l}, \mathbf{k} \in \mathbb{Z}^d$. Suppose that there is a positive constant C such that for every $\mathbf{n} \in \mathbb{N}^d$ and all $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, with $0 \leq k_j \leq n_j$ for $j = 1, 2, \ldots, d$, we have $\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\mathbf{p}-\mathbf{k}} \leq C a_{2\mathbf{n}-\mathbf{k}}$. Let*

$$\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x}) = e^{i\mathbf{l}\cdot\mathbf{x}} - \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} e^{i\mathbf{l}\cdot\mathbf{x}_{\mathbf{j}}} \widetilde{sk}_{\mathbf{n}}\left(\mathbf{x} - \mathbf{x}_{\mathbf{j}}\right).$$

Then for $\mathbf{l} \in \mathbb{Z}^d$, $\tilde{\mathbf{l}} = (|l_1|, \ldots, |l_d|)$,

$$|\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})| \leq \begin{cases} 4C\dfrac{a_{2\mathbf{n}-\tilde{\mathbf{l}}}}{a_{\mathbf{l}}}, & 0 < |\mathbf{l}| \leq |\mathbf{n}|, \\ |e^{i\mathbf{l}\cdot\mathbf{x}} - 1|, & \text{for } \ \mathbf{l} \equiv \mathbf{0} \bmod (2\mathbf{n}), \\ 4, & \text{for all } \mathbf{l}. \end{cases}$$

Proof: Let $\mu_{\mathbf{n},\mathbf{l}}(\mathbf{x})$ be the real part of $\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})$. For $\mathbf{l} \equiv \mathbf{0} \bmod (2\mathbf{n})$, by the Definition 4.1, Lemmas 3.2 and 5.1 we have

$$\mu_{\mathbf{n},\mathbf{l}}(\mathbf{x}) = \cos(\mathbf{l}\cdot\mathbf{x}) - \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} (\cos\left(\mathbf{l}\cdot\mathbf{x}_{\mathbf{j}}\right)) \widetilde{sk}_{\mathbf{n}}\left(\mathbf{x} - \mathbf{x}_{\mathbf{j}}\right) = \cos(\mathbf{l}\cdot\mathbf{x}) - 1.$$

For $\mathbf{l} \not\equiv \mathbf{0} \bmod (2\mathbf{n})$, using Lemma 5.2 we have

$$\mu_{\mathbf{n},\mathbf{l}}(\mathbf{x}) \;=\; \cos(\mathbf{l}\cdot\mathbf{x}) - \frac{\rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})} = \frac{\rho_{\mathbf{l}}(\mathbf{0})\cos(\mathbf{l}\cdot\mathbf{x}) - \rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})}.$$

Thus, as in the Theorem 3.6 we have $\rho_{\mathbf{l}}(\mathbf{x}) \le \rho_{\mathbf{l}}(\mathbf{0})$ for all $\mathbf{x}$ and for all $\mathbf{l} \in \mathbb{Z}^d$, then

$$|\mu_{\mathbf{n},\mathbf{l}}(\mathbf{x})| \;=\; \left|\frac{\rho_{\mathbf{l}}(\mathbf{0})\cos(\mathbf{l}\cdot\mathbf{x}) - \rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})}\right| \le \frac{\rho_{\mathbf{l}}(\mathbf{0}) + \rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})} \le \frac{2\rho_{\mathbf{l}}(\mathbf{0})}{\rho_{\mathbf{l}}(\mathbf{0})} = 2.$$

Suppose now that $0 < |\mathbf{l}| \le |\mathbf{n}|$ and let $\tilde{\mathbf{l}} = (|l_1|, \ldots, |l_d|)$. Thus using the hypothesis and the Remark 5.3, we obtain

$$|\mu_{\mathbf{n},\mathbf{l}}(\mathbf{x})| \;=\; \left|\frac{\rho_{\mathbf{l}}(\mathbf{0})\cos(\mathbf{l}\cdot\mathbf{x}) - \rho_{\mathbf{l}}(\mathbf{x})}{\rho_{\mathbf{l}}(\mathbf{0})}\right| \le \frac{2\rho_{\mathbf{l}}(\mathbf{0})}{2a_{\mathbf{l}}} \le 2C\frac{a_{2\mathbf{n}-\tilde{\mathbf{l}}}}{a_{\mathbf{l}}}$$

The imaginary part of $\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})$ is given by

$$\phi_{\mathbf{n},\mathbf{l}}(\mathbf{x}) = \sin\left(\mathbf{l}\cdot\mathbf{x}\right) - \sum_{\mathbf{j}\in\Omega_{\mathbf{n}}} \left(\sin\left(\mathbf{l}\cdot\mathbf{x}_{\mathbf{j}}\right)\right)\widetilde{sk}_{\mathbf{n}}\left(\mathbf{x}-\mathbf{x}_{\mathbf{j}}\right).$$

The estimate for the imaginary part is analogous to the real part. Considering the estimates obtained for $\mu_{\mathbf{n},\mathbf{l}}(\mathbf{x})$ and $\phi_{\mathbf{n},\mathbf{l}}(\mathbf{x})$, we obtain the desired estimate for $\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})$. □

Lemma 5.6. *Let K be a kernel as in Lemma 5.5. Then for each $1 \le p < \infty$, there is a positive constant C, depending only on p, such that*

$$\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}}^p |\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})|^p \le C \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p.$$

Proof: Since $a_{\mathbf{l}} \ge a_{\mathbf{k}} > 0$ if $|\mathbf{k}| \ge |\mathbf{l}|$, $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$, using Lemma 5.5 and taking

$\tilde{\mathbf{l}} = (|l_1|, \ldots, |l_d|)$ we have

$$\begin{aligned}
\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}}^p |\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})|^p &\le \sum_{0<|\mathbf{l}|\le|\mathbf{n}|} a_{\mathbf{l}}^p |\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})|^p + \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p |\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})|^p \\
&\le \sum_{0<|\mathbf{l}|\le|\mathbf{n}|} a_{\mathbf{l}}^p 4^p C^p \left(\frac{a_{2\mathbf{n}-\tilde{\mathbf{l}}}}{a_{\mathbf{l}}}\right)^p + \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p 4^p \\
&= 4^p C^p \sum_{0<|\mathbf{l}|\le|\mathbf{n}|} a_{2\mathbf{n}-\tilde{\mathbf{l}}}^p + 4^p \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p.
\end{aligned}$$

For each $\mathbf{l} \in \mathbb{Z}^d, \mathbf{l} \neq \mathbf{0}$, let $D_{\mathbf{l}} = \{\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d : |k_j| = |l_j|,\ 1 \le j \le d\}$. Then

$$\begin{aligned}
\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}}^p |\theta_{\mathbf{n},\mathbf{l}}(\mathbf{x})|^p &\le 4^p C^p \sum_{0<|\mathbf{l}|\le|\mathbf{n}|} \sum_{\mathbf{k}\in D_{\mathbf{l}}} a_{2\mathbf{n}-\mathbf{k}}^p + 4^p \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p \\
&\le 4^p C^p 2^d \sum_{0<|\mathbf{l}|\le|\mathbf{n}|} a_{2\mathbf{n}-\mathbf{l}}^p + 4^p \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p \\
&\le C_1 \left(\sum_{|\mathbf{n}|\le|\mathbf{j}|\le 3|\mathbf{n}|} a_{\mathbf{j}}^p + \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p \right) \le 2C_1 \sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^p,
\end{aligned}$$

completing the proof of the lemma. □

Theorem 5.7. *Let $|\cdot|$ be a norm on $\mathbb{R}^d$ and let K be a kernel given by*

$$K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}},$$

where $(a_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$ is a sequence that satisfies $a_{\mathbf{l}} = a_{\mathbf{k}}$ if $|\mathbf{l}| = |\mathbf{k}|$ and $a_{\mathbf{l}} \ge a_{\mathbf{k}} > 0$ if $|\mathbf{k}| \ge |\mathbf{l}|$, for $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$. Suppose that

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\mathbf{p}-\mathbf{k}} \le C a_{2\mathbf{n}-\mathbf{k}},$$

for all $\mathbf{n} \in \mathbb{N}^d$ and all $\mathbf{k} = (k_1, \ldots, k_d)$ with $0 \le k_j \le n_j$ for $j = 1, 2, \ldots, d$, where C is a positive constant that is independent of $\mathbf{n}$ and $\mathbf{k}$. Then for

$1 \le p \le 2 \le q \le \infty$, *with* $p^{-1} - q^{-1} \ge 2^{-1}$, *we have*

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le C \left(\sum_{|\mathbf{l}| \ge |\mathbf{n}|} a_{\mathbf{l}}^{qp(q-p)^{-1}} \right)^{p^{-1}-q^{-1}} .$$

Proof: Let $p \in \mathbb{R}$, $1 \le p \le 2$ and let p' such that $1/p + 1/p' = 1$. Given $f \in K * U_p$, $\phi \in U_p$ such that $f = K * \phi$, by Theorem 4.5,

$$\begin{aligned} \sigma_{\mathbf{n}}(f, \mathbf{x}) &= f(\mathbf{x}) - sk_{\mathbf{n}}(f, \mathbf{x}) \\ &= \int_{\mathbb{T}^d} \left(K(\mathbf{x} - \mathbf{y}) - \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} K(\mathbf{x_k} - \mathbf{y}) \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_k}) \right) \phi(\mathbf{y}) d\nu(\mathbf{y}) \\ &= \int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) d\nu(\mathbf{y}), \end{aligned}$$

where $\Phi_{\mathbf{n}}(\mathbf{x}, \mathbf{y}) = K(\mathbf{x}, \mathbf{y}) - \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} K(\mathbf{x_k} - \mathbf{y}) \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_k})$. Thus by Hölder inequality we have that

$$|f(\mathbf{x}) - sk_{\mathbf{n}}(f, \mathbf{x})| \le ||\phi||_p ||\Phi_{\mathbf{n}}(\mathbf{x}, \cdot)||_{p'}. \tag{11}$$

Since $1 \le p \le 2$, it follows from Hausdorff-Young inequality that

$$||\Phi_{\mathbf{n}}(\mathbf{x}, \cdot)||_{p'} \le \left(\sum_{\mathbf{l} \in \mathbb{Z}^d} |b_{\mathbf{l}}|^p \right)^{1/p},$$

where for $\mathbf{l} \in \mathbb{Z}^d$, $b_{\mathbf{l}} = \int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\mathbf{x}, \mathbf{y}) e^{-i\mathbf{l} \cdot \mathbf{y}} d\nu(\mathbf{y})$. By Lemma 3.1 we have that

$$\int_{\mathbb{T}^d} K(\mathbf{x} - \mathbf{y}) e^{-i\mathbf{l} \cdot \mathbf{y}} d\mathbf{y} = \sum_{\mathbf{j} \in \mathbb{Z}^d} a_{\mathbf{j}} e^{i\mathbf{j} \cdot \mathbf{x}} \int_{\mathbb{T}^d} e^{-i(\mathbf{l} + \mathbf{j}) \cdot \mathbf{y}} d\nu(\mathbf{y}) = a_{\mathbf{l}} e^{-i\mathbf{l} \cdot \mathbf{x}}$$

and in an analogous way

$$\int_{\mathbb{T}^d} \left(\sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} K(\mathbf{x_k} - \mathbf{y}) \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_k}) e^{-i\mathbf{l} \cdot \mathbf{y}} \right) d\nu(\mathbf{y}) = \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} a_{\mathbf{l}} e^{-i\mathbf{l} \cdot \mathbf{x_k}} \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_k}).$$

Thus

$$b_{\mathbf{l}} = a_{\mathbf{l}} \left(e^{-i\mathbf{l} \cdot \mathbf{x}} - \sum_{\mathbf{k} \in \Omega_{\mathbf{n}}} e^{-i\mathbf{l} \cdot \mathbf{x_k}} \widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_k}) \right) = a_{\mathbf{l}} \theta_{\mathbf{n}, -\mathbf{l}}(\mathbf{x}) = a_{-\mathbf{l}} \theta_{\mathbf{n}, -\mathbf{l}}(\mathbf{x}).$$

Using Lemma 5.6 we obtain

$$||\Phi_{\mathbf{n}}(\mathbf{x},\cdot)||_{p'} \leq \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{-\mathbf{l}}^p\,|\theta_{\mathbf{n},-\mathbf{l}}(\mathbf{x})|^p\right)^{1/p} \leq C\left(\sum_{|\mathbf{l}|\geq|\mathbf{n}|} a_{\mathbf{l}}^p\right)^{1/p}. \qquad (12)$$

For $\phi \in L^p(\mathbb{T}^d)$ we define

$$T\phi(\mathbf{x}) = \int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})d\nu(\mathbf{y}).$$

By inequalities (11) and (12) we conclude that T is a bounded operator from $L^p(\mathbb{T}^d)$ to $L^\infty(\mathbb{T}^d)$ and that

$$||T||_{p,\infty} \leq C\left(\sum_{|\mathbf{l}|\geq|\mathbf{n}|} a_{\mathbf{l}}^p\right)^{1/p}. \qquad (13)$$

By duality, T is bounded from $L^1(\mathbb{T}^d)$ to $L^{p'}(\mathbb{T}^d)$ and

$$||T||_{1,p'} \leq C\left(\sum_{|\mathbf{l}|\geq|\mathbf{n}|} a_{\mathbf{l}}^p\right)^{1/p}. \qquad (14)$$

Applying the Riesz-Thorin Interpolation Theorem we have $1 \leq (p_t^{-1}-q_t^{-1})^{-1} \leq p$ and

$$||T||_{p_t,q_t} \leq C\left(\sum_{|\mathbf{l}|\geq|\mathbf{n}|} a_{\mathbf{l}}^{q_t p_t (q_t-p_t)^{-1}}\right)^{p_t^{-1}-q_t^{-1}}.$$

If $1 \leq r \leq 2$, $2 \leq s \leq \infty$ and $1/r - 1/s \geq 1/2$, then there are $0 \leq t \leq 1$ and $1 \leq p \leq 2$ such that $1/r = 1 - t + t/p$ and $1/s = (1-t)/p'$, that is, $r = p_t$ and $s = q_t$. □

Lemma 5.8. *Let $a : [0,+\infty) \to \mathbb{R}$ be a decreasing and positive function and $|\cdot| = |\cdot|_p$ for some $1 \leq p \leq \infty$. For each $\mathbf{p} \in \mathbb{Z}^d$, let $a_{\mathbf{p}} = a(|\mathbf{p}|)$. Suppose that there is a constant $c_1 > 0$ such that for each $\mathbf{n} \in \mathbb{N}^d$,*

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\mathbf{p}} \leq c_1 a_{2\mathbf{n}}. \qquad (15)$$

Then there is a constant $c_2 > 0$ such that for each $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{k} \in \mathbb{Z}^d$ with $|\mathbf{k}| \leq |\mathbf{n}|$, we have

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{k}} \leq c_2 a_{2\mathbf{n}-\mathbf{k}}. \tag{16}$$

Proof: Fix $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. By Remark 5.3 it is enough to consider $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ with $|\mathbf{k}| \leq |\mathbf{n}|$ and $0 \leq k_j \leq n_j$, for each $j = 1, 2, \ldots, d$. Let $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{Z}^d$. For each $1 \leq j \leq d$, if

$$\psi_j(p_j) = \tilde{p}_j = \begin{cases} p_j - 1 & , \quad p_j > 0, \\ p_j & , \quad p_j \leq 0, \end{cases}$$

we define $\psi(\mathbf{p}) = \tilde{\mathbf{p}} = (\psi_1(p_1), \ldots, \psi_d(p_d))$, and ψ is well defined as a function from $\mathbb{Z}^d$ to $\mathbb{Z}^d$. As $|2n_j p_j - k_j| \geq |2n_j \tilde{p}_j|$ we have

$$|2\mathbf{np} - \mathbf{k}|_p \geq (|2n_1\tilde{p}_1|^p + \cdots + |2n_d\tilde{p}_d|^p)^{1/p} = |2\mathbf{n}\tilde{\mathbf{p}}|_p, \ 1 \leq p < \infty,$$

$$|2\mathbf{np} - \mathbf{k}|_\infty \geq \max\{|2n_1\tilde{p}_1|, \ldots, |2n_d\tilde{p}_d|\} = |2\mathbf{n}\tilde{\mathbf{p}}|_\infty.$$

Thus $|2\mathbf{np}-\mathbf{k}| \geq |2\mathbf{n}\tilde{\mathbf{p}}|$ and consequently $a_{2\mathbf{np}-\mathbf{k}} \leq a_{2\mathbf{n}\tilde{\mathbf{p}}}$. Since the cardinality of $\psi^{-1}(\{\mathbf{k}\})$ is at most 2^d for all $\mathbf{k} \in \mathbb{Z}^d$, by (15)

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{k}} \leq \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{n}\tilde{\mathbf{p}}} \leq 2^d \sum_{\tilde{\mathbf{p}}\in\mathbb{Z}^d} a_{2\mathbf{n}\tilde{\mathbf{p}}} \leq 2^d c_1 a_{2\mathbf{n}} \leq 2^d c_1 a_{2\mathbf{n}-\mathbf{k}},$$

and this concludes the proof. □

The next result is consequence of Theorem 5.7 and Lemma 5.8.

Corollary 5.9. *Let $a : [0, +\infty) \to \mathbb{R}$ be a decreasing and positive function and $|\cdot| = |\cdot|_p$ for some $1 \leq p \leq \infty$. For each $\mathbf{p} \in \mathbb{Z}^d$ let $a_{\mathbf{p}} = a(|\mathbf{p}|)$. Consider the kernel K given by*

$$K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}},$$

such that

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}} \leq C a_{2\mathbf{n}},$$

where C is a positive constant independent of $\mathbf{n} \in \mathbb{N}^d$. Then there is a positive constant $\overline{C}$, such that for each $1 \le p \le 2 \le q \le \infty$, with $p^{-1} - q^{-1} \ge 2^{-1}$ and all $\mathbf{n} \in \mathbb{N}^d$, we have

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le \overline{C} \left(\sum_{|\mathbf{l}| \ge |\mathbf{n}|} a_{\mathbf{l}}^{qp(q-p)^{-1}} \right)^{p^{-1}-q^{-1}}.$$

6 Approximation of finitely differentiable functions

Theorem 6.1. *For $\gamma \in \mathbb{R}$, $\gamma > d$, let*

$$K(\mathbf{x}) = \sum_{\mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} |\mathbf{l}|^{-\gamma} e^{i\mathbf{l}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{T}^d,$$

where $|\cdot| = |\cdot|_2$ or $|\cdot| = |\cdot|_\infty$. For $n \in \mathbb{N}$, let $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^d$. Then, for $1 \le p \le 2 \le q \le \infty$, with $1/p - 1/q \ge 1/2$, there is a positive constant $C_{p,q}$, independent of $n \in \mathbb{N}$, such that

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le C_{p,q} n^{-\gamma + d(1/p - 1/q)}. \tag{17}$$

Proof: Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Using the function $f(x) = (x-1)^\alpha / x^\alpha$, $x \ge 2$ we obtain

$$(j-1)^{-\alpha} \le 2^\alpha j^{-\alpha}, \ j \ge 2. \tag{18}$$

Fix $\mathbf{n} = (n, n, \ldots, n)$ and for each $j \in \mathbb{N}$ let $B_j = \{\mathbf{l} \in \mathbb{Z}^d : j - 1 \le |\mathbf{l}| < j\}$. Then $\mathbb{Z}^d = \bigcup_{j=1}^\infty B_j$. If $\mathbf{p} \in B_j$, then $j - 1 \le |\mathbf{p}| < j$ and thus $j^{-\gamma} < |\mathbf{p}|^{-\gamma} \le (j-1)^{-\gamma}$. Let $a_{\mathbf{l}} = |\mathbf{l}|^{-\gamma}$ for $\mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and $a_{\mathbf{0}} = 0$. We have $\dim \mathcal{H}_l \asymp \dim \mathcal{H}_l^* \asymp l^{d-1}$ and then the cardinality of B_j satisfies

$$\#B_j \le C j^{d-1}, \ j \in \mathbb{N}, \tag{19}$$

where C is a positive constant independent of j. Since $\mathbf{2np} = 2n\mathbf{p}$, by (18) and (19)

$$\begin{aligned}\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{\mathbf{2np}} &= \sum_{j=2}^{\infty}\sum_{\mathbf{p}\in B_j}|2n\mathbf{p}|^{-\gamma} \leq \sum_{j=2}^{\infty}\sum_{\mathbf{p}\in B_j}(2n)^{-\gamma}(j-1)^{-\gamma}\\ &\leq C\sum_{j=2}^{\infty}(2n)^{-\gamma}j^{d-1}(j-1)^{-\gamma} \leq 2^{\gamma}C(2n)^{-\gamma}\sum_{j=2}^{\infty}j^{d-1-\gamma}.\end{aligned}$$

Since $\gamma > d$, then $d-1-\gamma < 0$ and $\sum_{j=2}^{\infty} j^{d-1-\gamma} \leq \int_1^{\infty} t^{d-1-\gamma}dt$. Thus, since $d-\gamma < 0$,

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{\mathbf{2np}} \leq 2^{\gamma}C(2n)^{-\gamma}\lim_{m\to\infty}\int_1^m t^{d-1-\gamma}dt = \frac{2^{\gamma}C|\mathbf{1}|^{\gamma}}{\gamma-d}a_{\mathbf{2n}} = C_1 a_{\mathbf{2n}}. \quad (20)$$

Therefore the hypothesis of Corollary 5.9 is satisfied.

Let $r = p^{-1} - q^{-1}$ and $s = r^{-1}$. Then using (18) and (19)

$$\sum_{|\mathbf{l}|\geq|\mathbf{n}|}(a_{\mathbf{l}})^s \leq \sum_{j=|\mathbf{n}|+1}^{\infty}\sum_{\mathbf{l}\in B_j}(j-1)^{-s\gamma} \leq C\sum_{j=|\mathbf{n}|}^{\infty}(j+1)^{d-1}j^{-s\gamma} \leq 2^{d-1}C\sum_{j=|\mathbf{n}|}^{\infty}j^{d-1-s\gamma}.$$

We have $1 \leq p \leq 2$ and thus $r = 1/p - 1/q \leq 1$ and $s \geq 1$. Then $d-1-s\gamma < 0$ and therefore $j^{d-1-s\gamma} \leq \int_{j-1}^{j} t^{d-1-s\gamma}dt$. We obtain that

$$\begin{aligned}\sum_{|\mathbf{l}|\geq|\mathbf{n}|}(a_{\mathbf{l}})^s &\leq 2^{d-1}C\sum_{j=|\mathbf{n}|}^{\infty}\int_{j-1}^{j}t^{d-1-s\gamma}dt = -2^{d-1}C\frac{(|\mathbf{n}|-1)^{d-s\gamma}}{d-s\gamma}\\ &\leq \frac{2^{d-1}C}{s\gamma-d}|\mathbf{n}|^{d-s\gamma} = C_2|\mathbf{n}|^{d-s\gamma}.\end{aligned}$$

Applying the Corollary 5.9 we have

$$\sup_{f\in K*U_p}||f - sk_{\mathbf{n}}(f,\cdot)||_q \leq C_3\left(\sum_{|\mathbf{l}|\geq|\mathbf{n}|}(a_{\mathbf{l}})^s\right)^r \leq C_4|\mathbf{n}|^{-\gamma+d(p^{-1}-q^{-1})},$$

for $1 \leq p \leq 2 \leq q \leq \infty$, with $p^{-1} - q^{-1} \geq 2^{-1}$. □

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