LIMIT CYCLES BIFURCATING FROM DISCONTINUOUS POLYNOMIAL PERTURBATIONS OF HIGHER DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS

JAUME LLIBRE[1], DOUGLAS D. NOVAES[2] AND IRIS O. ZELI[2]

Abstract. We study the periodic solutions bifurcating from periodic orbits of linear differential systems $x' = Mx$, when they are perturbed inside a class of discontinuous piecewise polynomial differential systems with two zones. More precisely, we study the periodic solutions of the differential system

$$x' = Mx + \varepsilon F_1^n(x) + \varepsilon^2 F_2^n(x),$$

in $\mathbb{R}^{d+2}$ where ε is a small parameter, M is a $(d+2)\times(d+2)$ matrix having one pair of pure imaginary conjugate eigenvalues, m zeros eigenvalues, and $d-m$ non–zero real eigenvalues.

For solving this problem we need to extend the averaging theory for studying periodic solutions to a new class of non–autonomous $d+1$-dimensional discontinuous piecewise smooth differential system.

1. Introduction

The analysis of discontinuous piecewise smooth differential systems has recently a large and fast growth due to its applications in several areas of the knowledge. Such systems can model various phenomena in control systems (see [2]), impact on mechanical systems (see [3]), economy (see [12]), biology (see [13]), nonlinear oscillations (see [21]), neuroscience (see [6, 9, 22]), ...

The occurrence of limit cycles in differential systems have been used to model the behavior of many real process in different situations. The first studies on this subject considered smooth differential systems and, since then, many contributions have been made in this direction (see [11] and the references quoted therein). Recently the study of limit cycles has also been considered for either continuous (see, for instance, [1, 15, 18]) or discontinuous piecewise smooth differential systems (see, for instance, [10, 14, 16, 19, 20]).

2010 *Mathematics Subject Classification.* 34A36, 34C25, 34C29, 37G15.

Key words and phrases. limit cycle, averaging method, periodic orbit, polynomial differential system, nonsmooth polynomial differential systems, nonsmooth dynamical system, Filippov system.

One of the main tools used for determining limit cycles is the averaging theory. This theory was exhaustively used to deal with smooth differential systems. In [1] the averaging theory was extended, via topological methods, for studying periodic orbits of continuous (nonsmooth) differential systems. After in [18] the results of [1] were generalized to continuous differential systems with higher order perturbation.

Recently in [16] the averaging theory was extended for detecting periodic orbits of a class of discontinuous piecewise smooth differential systems. This generalization has already been used in some works, for example see [19, 23].

In this paper we are interested in studying the number of limit cycles bifurcating from the periodic orbits of a linear differential system $x' = Mx$, where M is a $(d+2)\times(d+2)$ matrix having one pair of pure imaginary conjugate eigenvalues, m zeros eigenvalues, and $d-m$ real eigenvalues. We focus our attention when this system is perturbed up to order 2 in the small parameter ε inside a class of discontinuous piecewise polynomial functions having two zones. In order to solve this problem we adapt the averaging theory for studying the periodic solutions of a class of non–autonomous $d+1$-dimensional discontinuous piecewise smooth differential system.

2. Statements of the main results

2.1. **Advances on averaging theory.** In this subsection we improve the averaging theory of first and second order to study the limit cycles of a class of discontinuous piecewise smooth differential systems.

Let D be an open bounded subset of $\mathbb{R}^{d+1}$ and for a positive real number T we consider the $\mathcal{C}^3$ differentiable functions $F_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}^{d+1}$ for $i = 0, 1, 2$, and $R^{\pm} : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{d+1}$ where $\mathbb{S}^1 \equiv \mathbb{R}/(\mathbb{Z}T)$. Thus we define the following T-periodic *discontinuous piecewise smooth differential system*

$$\mathbf{x}' = \begin{cases} F^{+}(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ F^{-}(\theta, \mathbf{x}, \varepsilon) & \text{if} \quad \phi \leq \theta \leq T, \end{cases} \tag{1}$$

where the prime denotes derivative with respect to the variable $\theta \in \mathbb{S}^1$, and

$$F^{\pm}(\theta, \mathbf{x}, \varepsilon) = F_0^{\pm}(\theta, \mathbf{x}) + \varepsilon F_1^{\pm}(\theta, \mathbf{x}) + \varepsilon^2 F_2^{\pm}(\theta, \mathbf{x}) + \varepsilon^3 R^{\pm}(\theta, \mathbf{x}, \varepsilon),$$

with $\mathbf{x} \in D$. The set of discontinuity of system (1) is given by $\Sigma = \{\theta = 0\} \cup \{\theta = \phi\}$.

For $\mathbf{z} \in D$, let $\varphi(\theta, \mathbf{z})$ be the solution of the unperturbed system

$$\mathbf{x}' = F_0(\theta, \mathbf{x}), \tag{2}$$

such that $\varphi(0,\mathbf{z})=\mathbf{z}$, where

$$F_0(\theta,\mathbf{x})=\begin{cases} F_0^+(\theta,\mathbf{x}) & \text{if} \quad 0\le\theta\le\phi,\\ F_0^-(\theta,\mathbf{x}) & \text{if} \quad \phi\le\theta\le T.\end{cases}$$

Clearly

$$\varphi(\theta,\mathbf{z})=\begin{cases} \varphi^+(\theta,\mathbf{z}) & \text{if} \quad 0\le\theta\le\phi,\\ \varphi^-(\theta,\mathbf{z}) & \text{if} \quad \phi\le\theta\le T,\end{cases}$$

where $\varphi^\pm(\theta,\mathbf{z})$ are the solutions of the systems

$$\mathbf{x}'=F_0^\pm(\theta,\mathbf{x}), \tag{3}$$

such that $\varphi^\pm(0,\mathbf{z})=\mathbf{z}$.

We assume that there exists a manifold $\mathcal{Z}$ embedded in D such that the solutions starting in $\mathcal{Z}$ are all T-periodic solutions. More precisely, for $p=d+1$ and $q\le p$, let $\sigma:\overline{V}\to\mathbb{R}^{p-q}$ be a $\mathcal{C}^3$ function being V an open and bounded subset of $\mathbb{R}^q$, and let

$$\mathcal{Z}=\{\mathbf{z}_\nu=(\nu,\sigma(\nu)):\ \nu\in\overline{V}\}. \tag{4}$$

We shall assume that

(H) $\mathcal{Z}\subset D$ and for each $\mathbf{z}_\nu$ the unique solution $\varphi(\theta,\mathbf{z}_\nu)$ such that $\varphi(0,\mathbf{z}_\nu)=\mathbf{z}_\nu$ is T-periodic.

For $\mathbf{z}\in D$ we consider the first order variational equations of systems (3) along the solution $\varphi^\pm(\theta,\mathbf{z})$, that is

$$Y'=D_{\mathbf{x}}F_0^\pm(\theta,\varphi^\pm(\theta,\mathbf{z}))\,Y. \tag{5}$$

Denote by $Y^\pm(\theta,\mathbf{z})$ a fundamental matrix of the differential system (5).

Let $\xi:\mathbb{R}^q\times\mathbb{R}^{p-q}\to\mathbb{R}^q$ and $\xi^\perp:\mathbb{R}^q\times\mathbb{R}^{p-q}\to\mathbb{R}^{p-q}$ be the orthogonal projections onto the first q coordinates and onto the last $p-q$ coordinates, respectively. For a point $\mathbf{z}\in$ D denote $\mathbf{z}=(u,v)\in\mathbb{R}^q\times\mathbb{R}^{p-q}$. Thus we define the averaged functions $f_1,f_2:\overline{V}\to\mathbb{R}^q$ as

$$\begin{aligned} f_1(\nu) &= \xi g_1(\mathbf{z}_\nu),\\ f_2(\nu) &= 2\frac{\partial\xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu)+\frac{\partial^2\xi g_0}{\partial v^2}(\mathbf{z}_\nu)\gamma(\nu)^2+2\xi g_2(\mathbf{z}_\nu),\end{aligned} \tag{6}$$

where

$$\gamma(\nu)=-\Delta_\nu^{-1}\xi^\perp g_1(\mathbf{z}_\nu), \tag{7}$$

$$g_i(\mathbf{z}) = y_i^+(\phi, \mathbf{z}) - y_i^-(\phi - T, \mathbf{z}), \text{ for } j = 0, 1, 2 \text{ and} \tag{8}$$

$$\begin{aligned}
y_0^\pm(\theta, \mathbf{z}) &= \varphi^\pm(\theta, \mathbf{z}), \\
y_1^\pm(\theta, \mathbf{z}) &= Y^\pm(\theta, \mathbf{z}) \int_0^\theta Y^\pm(s, \mathbf{z})^{-1} F_1^\pm(s, \varphi^\pm(s, \mathbf{z})) ds, \\
y_2^\pm(\theta, \mathbf{z}) &= Y^\pm(\theta, \mathbf{z}) \int_0^\theta Y^\pm(s, \mathbf{z})^{-1} \Bigg(2F_2^\pm(s, \varphi^\pm(s, \mathbf{z})) + \\
& \qquad 2\frac{\partial F_1^\pm}{\partial \mathbf{x}}(s, \varphi(s, \mathbf{z})) y_1^\pm(s, \mathbf{z}) + \frac{\partial^2 F_0^\pm}{\partial \mathbf{x}^2}(s, \varphi^\pm(s, \mathbf{z})) y_1^\pm(s, \mathbf{z})^2 \Bigg) ds.
\end{aligned} \tag{9}$$

The $(p-q) \times (p-q)$ matrix Δ_ν is defined in the statement of the next theorem.

Our main result on the periodic solutions of system (1) is the following.

Theorem 1. *In addition to hypothesis (H) we assume that for any $\nu \in \overline{V}$ the matrix $Y^+(\phi, \nu) - Y^-(\phi - T, \nu)$ has in the upper right corner the null $q \times (p-q)$ matrix, and in the lower right corner has the $(p-q) \times (p-q)$ matrix Δ_ν with* $\det(\Delta_\nu) \neq 0$. *Then the following statements hold.*

- (a) *If there exists $\nu^* \in V$ such that $f_1(\nu^*) = 0$ and $\det(f_1'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(\theta, \varepsilon)$ of system (1) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.*
- (b) *Assume that $f_1 \equiv 0$. If there exists $\nu^* \in V$ such that $f_2(\nu^*) = 0$ and $\det(f_2'(\nu^*)) \neq 0$, then for $|\varepsilon| > 0$ sufficiently small there exists a T–periodic solution $\mathbf{x}(\theta, \varepsilon)$ of system (1) such that $\mathbf{x}(0, \varepsilon) \to \mathbf{z}_{\nu^*}$ as $\varepsilon \to 0$.*

Theorem 1 is proved in section 4. The following result is an immediate consequence of Theorem 1.

Corollary 2. *Assume the hypothesis (H) and that $q = p$, in this case $\mathcal{Z} = \overline{V} \subset D$ is a compact bounded p–dimensional manifold. Then statements (a) and (b) of Theorem 1 hold by taking $f_1 = g_1$ and $f_2 = 2g_2$.*

2.2. **Perturbations of higher dimensional linear systems.** Consider a $(d+2) \times (d+2)$ matrix M given by

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \widetilde{M} \end{pmatrix}.$$

When $0 < m < d$ assume that $\widetilde{M}$ is the diagonal matrix diag$(\mu_1, \mu_2, \ldots, \mu_d)$ with $\mu_1 = \ldots = \mu_m = 0$ and $\mu_{m+1} \neq 0, \ldots, \mu_d \neq 0$. If $m = 0$, then $\widetilde{M}$ is a diagonal matrix with all entries distinct from zero, and if $m = d$ we assume that $\widetilde{M}$ is the null matrix.

Let $L_1 = \{(1, 0, z) : x \geq 0, \mathrm{z} \in \mathbb{R}^d\}$ and $L_2 = \{(\lambda\cos\phi, \lambda\sin\phi, z) : \lambda \geq 0, \mathrm{z} \in \mathbb{R}^d\}$ be two half–hyperplanes of $\mathbb{R}^{d+2}$ sharing the boundary $\{0, 0, z) : \mathrm{z} \in \mathbb{R}^d\}$. The set $\Sigma = L_1 \cup L_2$ splits $D \subset \mathbb{R}^{d+2}$ in 2 disjoint open sectors, namely C^+ and C^- (see Figure 2.2).

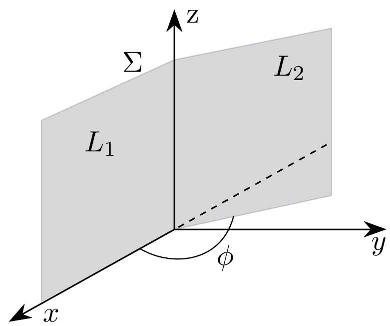

Figure 1. Set of discontinuity Σ.

We will denote by X_λ and Y_λ two polynomials of degree n in the variables $x, y \in \mathbb{R}$ and $\mathrm{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$, more precisely

$$X_\lambda(x, y, \mathrm{z}) = \sum_{i+j+k_1+\ldots+k_d=0}^{n} \lambda_{ijk_1\ldots k_d}\, x^i y^j z_1^{k_1} \ldots z_d^{k_d}, \text{ and}$$

$$Y_\lambda(x, y, \mathrm{z}) = \sum_{i+j+k_1+\ldots+k_d=0}^{n} \lambda_{ijk_1\ldots k_d}\, x^i y^j z_1^{k_1} \ldots z_d^{k_d},$$

for $\lambda_{ijk_1\ldots k_d} \in \mathbb{R}$ and $i, j, k_1, \ldots, k_d \in \mathbb{R}_+$. Then, take

$$X^{\pm} = (X_{a^\pm}, X_{b^\pm}, X_{c_1^\pm}, \ldots, X_{c_d^\pm}), \quad Y^{\pm} = (Y_{\alpha^\pm}, Y_{\beta^\pm}, Y_{\gamma_1^\pm}, \ldots, Y_{\gamma_d^\pm}),$$

and let $\mathcal{X}(x, y, \mathrm{z})$ and $\mathcal{Y}(x, y, \mathrm{z})$ be polynomial vector fields defined by

$$\mathcal{X}(x, y, \mathrm{z}) = X^{\pm}(x, y, \mathrm{z}) \quad \text{if} \quad (x, y, \mathrm{z}) \in C^{\pm},$$
$$\mathcal{Y}(x, y, \mathrm{z}) = Y^{\pm}(x, y, \mathrm{z}) \quad \text{if} \quad (x, y, \mathrm{z}) \in C^{\pm}.$$

Now consider the discontinuous piecewise polynomial differential systems

$$(\dot{x}, \dot{y}, \dot{\mathrm{z}}) = M\,(x, y, \mathrm{z}) + \varepsilon\mathcal{X}\,(x, y, \mathrm{z}) + \varepsilon^2\mathcal{Y}\,(x, y, \mathrm{z})\,, \tag{10}$$

where $x, y \in \mathbb{R}$ and $\mathrm{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{R}^d$. The dot denotes derivative with respect to the time t, and Σ denotes the set of discontinuity for system (10).

Denote by $N_i(m,n,\phi)$ the maximum number of limit cycles of system (10) that can be detected using averaging theory of order i when $|\varepsilon| \neq 0$ is sufficiently small.

Theorem 3. *Assume* $0 \leq m \leq d$ *and* $\phi \neq \pi$. *Then*

(*a*) $N_1(m,n,\phi) = n^{m+1}$,

(*b*) $2n(2n-1)^m \leq N_2(m,n,\phi) \leq (2n)^{m+1}$ *and*

(*c*) $N_2(1,n,\phi) = (2n)^2$.

Theorem 3 generalizes the particular cases $m = d$ of [19]. The lower bound given in statement (*b*) of Theorem 3 is not optimal and can be improved in some cases. Indeed, for $0 \leq \ell \leq m$ let P_ℓ be a polynomial of degree n and Q_ℓ be a polynomial of degree $2n$ in the variables (r, z) with $\mathrm{z} = (z_1, z_2, \ldots, z_m)$. Consider the polynomial system

$$
\begin{aligned}
& rP_0(r,\mathrm{z}) + Q_0(r,\mathrm{z}) = 0, \\
& \vdots \\
& rP_m(r,\mathrm{z}) + Q_m(r,\mathrm{z}) = 0.
\end{aligned}
\tag{11}
$$

The above system can be used to improve the lower bound of $N_2(m,n,\phi)$ as following.

Theorem 4. *Suppose that there exist polynomials* P_ℓ *and* Q_ℓ *such that system* (11) *has* N *isolated solutions. Then there exist polynomials* $X_{\lambda^\pm}$ *and* $Y_{\omega^\pm}$ *for* $\lambda = a, b, c_\rho$, $\omega = \alpha, \beta, \gamma_\rho$, *and* $\rho = 1, \ldots, d$ *such that system* (10) *has at least* N *limit cycles.*

Corollary 5. *Let* N_0 *be the maximum number of isolated solutions that system* (11) *can have for any choose of polynomials* P_ℓ, *and* Q_ℓ. *Then* $N_2(m,n,\phi) \geq N_0$.

Theorems 3 and 4 are proved in section 5.

If $\phi = \pi$ we note that the number of limit cycles can decreases and this fact is presented below.

Theorem 6. *Assume* $0 \leq m \leq d$ *and* $\phi = \pi$. *Then*

(*a*) $N_1(m,n,\pi) = n^{m+1}$ *and*

(*b*) $N \leq N_2(m,n,\pi) \leq (2n)^{m+1}$ *where* $N = (2n-1)^{m+1}$ *if* n *is odd, and* $N = (2n-2)(2n-1)^m$ *if* n *is even.*

Theorem 6 is proved in section 6.

When $\phi = 2\pi$, system (10) is continuous. In this case $\mathcal{X}(x, y, \mathrm{z}) = X^+(x, y, \mathrm{z})$ and $\mathcal{Y}(x, y, \mathrm{z}) = Y^+(x, y, \mathrm{z})$. So we get the following result.

Theorem 7. *Assume that* $0 \le m \le d$ *and* $\phi = 2\pi$. *Then*

(*a*) $N_1(m, n, 2\pi) = n^m(n-1)/2$ *for all* $m \neq 0$, *and*

$$N_1(0, n, 2\pi) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd,} \\ \dfrac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

(*b*) $N \le N_2(0, n, 2\pi) \le 2n$, *where* $N = n - 1$ *if* n *is odd, and* $N = n$ *if* n *is even.*

Theorem 7 generalizes the particular cases $m = d = 0$ and $m = d = 1$ of [5] (see Theorems 2 and 3). Moreover, statement (*a*) of Theorem 7 also generalizes Theorem 1 of [19] when $m = d$. We prove Theorem 7 in section 7.

3. Preliminary results

In this section we present some preliminaries results that we shall need in sections 5, 6 and 7. In subsection 3.1 we present a change of coordinates so that system (10) reads as system (1), and in subsection 3.2 we construct the averaging functions f_1 and f_2 for system (10), defined in (6). Finally, in subsection 3.3 we present some trigonometric relations that will be used in the calculus of the zeros of the functions f_1 and f_2.

3.1. **Setting the problem.** Let $x, y \in \mathbb{R}$ and $\mathrm{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$. Using the change of variables

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \tag{12}$$

with $r \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/2\pi$, system (10) becomes

$$\left(\dot\theta, \dot r, \dot{\mathrm{z}}\right) = (1, 0, \widetilde{M}\mathrm{z}) + \varepsilon A(\theta, r, \mathrm{z}) + \varepsilon^2 B(\theta, r, \mathrm{z}), \tag{13}$$

where $A, B : \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{R}^d \to \mathbb{R}^{d+2}$ are piecewise smooth functions given by

$$A = \begin{cases} A^+ & \text{if} \quad 0 \le \theta \le \phi, \\ A^- & \text{if} \quad \phi \le \theta \le 2\pi, \end{cases} \qquad \text{and} \quad B = \begin{cases} B^+ & \text{if} \quad 0 \le \theta \le \phi, \\ B^- & \text{if} \quad \phi \le \theta \le 2\pi, \end{cases}$$

where

$$A^{\pm}(\theta,r,\mathrm{z}) = \quad (A_1^{\pm}(\theta,r,\mathrm{z}),\ldots,A_{d+2}^{\pm}(\theta,r,\mathrm{z})),$$
$$B^{\pm}(\theta,r,\mathrm{z}) = \quad (B_1^{\pm}(\theta,r,\mathrm{z}),\ldots,B_{d+2}^{\pm}(\theta,r,\mathrm{z})),$$

with

$$\begin{aligned}
A_1^{\pm} =& \quad \frac{1}{r}\big(X_{b^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\cos\theta - X_{a^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\sin\theta\big),\\
B_1^{\pm} =& \quad \frac{1}{r}\big(Y_{\beta^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\cos\theta - Y_{\alpha^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\sin\theta\big),\\
A_2^{\pm} =& \quad X_{a^{\pm}}\,(r\cos\theta,r\sin\theta,z)\cos\theta + X_{b^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\sin\theta,\\
B_2^{\pm} =& \quad Y_{\alpha^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\cos\theta + Y_{\beta^{\pm}}\,(r\cos\theta,r\sin\theta,\mathrm{z})\sin\theta,\\
A_{\ell+2}^{\pm} =& \quad X_{c_\ell^{\pm}}(r\cos\theta,r\sin\theta,\mathrm{z}),\\
B_{\ell+2}^{\pm} =& \quad Y_{\gamma_\ell^{\pm}}(r\cos\theta,r\sin\theta,\mathrm{z}),
\end{aligned} \tag{14}$$

for $1\leq \ell\leq d$. Clearly the discontinuity Σ is now given by

$$\Sigma = \big\{(0,r,\mathrm{z}) : r\in\mathbb{R}_+,\mathrm{z}\in\mathbb{R}^d\big\}\cup\big\{(\phi,r,\mathrm{z}) : r\in\mathbb{R}_+,\mathrm{z}\in\mathbb{R}^d\big\}.$$

Taking the angle θ as the new time, system (13) reads

$$\begin{aligned}
r' = \frac{\dot r}{\dot\theta} &= \frac{\varepsilon A_2(\theta,r,\mathrm{z}) + \varepsilon^2 B_2(\theta,r,\mathrm{z})}{1+\varepsilon A_1(\theta,r,\mathrm{z}) + \varepsilon^2 B_1(\theta,r,\mathrm{z})},\\
z_\ell' = \frac{\dot z_\ell}{\dot\theta} &= \frac{\mu_\ell z_\ell + \varepsilon A_{\ell+2}(\theta,r,\mathrm{z}) + \varepsilon^2 B_{\ell+2}(\theta,r,\mathrm{z})}{1+\varepsilon A_1(\theta,r,\mathrm{z}) + \varepsilon^2 B_1(\theta,r,\mathrm{z})},
\end{aligned} \tag{15}$$

for $1\leq \ell\leq d$. Note that now the prime denotes derivative with respect to the independent variable θ.

Expanding system (15) in Taylor series around $\varepsilon = 0$, it can be written as system (1) by taking $\mathbf{x} = (r,\mathrm{z})\in D\subset\mathbb{R}_+\times\mathbb{R}^d$ and

$$F_j^{\pm}(\theta,r,\mathrm{z}) = (F_{j0}^{\pm}(\theta,r,\mathrm{z}),\ldots,F_{jd}^{\pm}(\theta,r,\mathrm{z})), \quad \text{for} \quad j=0,1,2, \tag{16}$$

where

$$
\begin{aligned}
F^{\pm}_{0\ell}(\theta,r,\mathrm{z}) =&\quad 0,\\
F^{\pm}_{0\omega}(\theta,r,\mathrm{z}) =&\quad \mu_\omega z_\omega,\\
F^{\pm}_{1\ell}(\theta,r,\mathrm{z}) =&\quad A^{\pm}_{\ell+2}(\theta,r,\mathrm{z}),\\
F^{\pm}_{1\omega}(\theta,r,\mathrm{z}) =&\quad A^{\pm}_{\omega+2}(\theta,r,\mathrm{z})-\mu_\omega z_\omega A^{\pm}_{1}(\theta,r,\mathrm{z}),\\
F^{\pm}_{2\ell}(\theta,r,\mathrm{z}) =&\quad B^{\pm}_{\ell+2}(\theta,r,\mathrm{z})-A^{\pm}_{1}(\theta,r,\mathrm{z})A^{\pm}_{\ell+2}(\theta,r,\mathrm{z}),\\
F^{\pm}_{2\omega}(\theta,r,\mathrm{z}) =&\quad B^{\pm}_{\omega+2}(\theta,r,\mathrm{z})+\mu_\omega z_\omega\big(A^{\pm}_{1}(\theta,r,\mathrm{z})\big)^2\\
&-A^{\pm}_{1}(\theta,r,\mathrm{z})A^{\pm}_{\omega+2}(\theta,r,\mathrm{z})-\mu_\omega z_\omega B^{\pm}_{1}(\theta,r,\mathrm{z}),
\end{aligned}
\tag{17}
$$

for $0\le\ell\le m$ and $m+1\le\omega\le d$.

When $m=d$ the functions $F^{\pm}_{j\omega}$, for $j=0,1,2$, do not be considered.

3.2. **Construction of the averaging functions.** Now we shall use the notations introduced in subsection 2.1. Since the unperturbed system (2) is continuous, we have $\varphi^+(\theta,\mathbf{z})=\varphi^-(\theta,\mathbf{z})$. Therefore when $0\le m<d$ the solution of system (2) is given by

$$\varphi(\theta,\mathbf{z})=(r,z_1,\ldots,z_m,e^{\theta}z_{m+1},\ldots,e^{\theta}z_d),$$

for $\mathbf{z}=(r,\mathrm{z})=(r,z_1,\ldots,z_d)$. Note that if $\mathbf{z}_\nu=(r,z_1,\ldots,z_m,0,\ldots,0)$ then $\varphi(\theta,\mathbf{z}_\nu)=\mathbf{z}_\nu$ for every $\theta\in\mathbb{S}^1$. Then taking an open bounded subset $V\subset\mathbb{R}^{m+1}$ and the zero function $\sigma:\overline{V}\to\mathbb{R}^{d-m}$, the manifold $\mathcal{Z}$, defined in (4), becomes

$$\mathcal{Z}=\{\mathbf{z}_\nu=(\nu,0)\in\mathbb{R}^{d+1}:\nu=(r,z_1,\ldots,z_m)\in\overline{V}\}.$$

For $\mathbf{z}\in D$ a fundamental matrix of system (5) is

$$Y(\theta,\mathbf{z})=\begin{pmatrix}\mathrm{Id}_{1+m} & 0\\ 0 & \Delta\end{pmatrix},$$

where Id_{1+m} is the $(1+m)\times(1+m)$ identity matrix, and Δ is the diagonal matrix $\mathrm{diag}(e^{\mu_{m+1}\theta},\ldots,e^{\mu_d\theta})$. Since $Y(\theta,\mathbf{z})$ does not depend of $\mathbf{z}$ we denote $Y(\theta,\mathbf{z})=Y(\theta)$. Then we have

$$Y(\phi)-Y(\phi-2\pi)=\begin{pmatrix}0 & 0\\ 0 & \Delta_\nu\end{pmatrix},$$

where $\Delta_\nu = \operatorname{diag}\big(e^{\mu_{m+1}\phi}(1-e^{-\mu_{m+1}2\pi}),\ldots,e^{\mu_d\phi}(1-e^{-\mu_d 2\pi})\big)$.

According to the notation introduced in Theorem 1 we have $p=d+1$ and $p-q=d-m$, with $q=m+1$. Since $\mathcal{Z}$ has dimension $m+1$, we consider the projections $\xi:\mathbb{R}^{m+1}\times\mathbb{R}^{d-m}\to\mathbb{R}^{m+1}$ and $\xi^\perp:\mathbb{R}^{m+1}\times\mathbb{R}^{d-m}\to\mathbb{R}^{d-m}$, with $u=(r,z_1,\ldots,z_m)\in\mathbb{R}^{m+1}$ and $v=(z_{m+1},\ldots,z_d)\in\mathbb{R}^{d-m}$.

From (9) we have $y_1(\theta,\mathbf{z})=\big(y_{10}(\theta,\mathbf{z}),\ldots,y_{1d}(\theta,\mathbf{z})\big)$ where

$$
\begin{aligned}
y_{1\ell}^{\pm}(\theta,\mathbf{z}) &= \int_0^\theta A_{\ell+2}^{\pm}(s,\varphi(s,\mathbf{z}))ds,\\
y_{1\omega}^{\pm}(\theta,\mathbf{z}) &= \int_0^\theta e^{\mu_\omega(\theta-s)}\big(A_{\omega+2}^{\pm}(s,\varphi(s,\mathbf{z}))-\mu_\omega z_\omega A_1^{\pm}(s,\varphi(s,\mathbf{z}))\big)ds,
\end{aligned}
\tag{18}
$$

for $0\le\ell\le m$ and $m+1\le\omega\le d$. Moreover, from (8) we have $g_1(\mathbf{z})=\big(g_{10}(\mathbf{z}),\ldots,g_{1d}(\mathbf{z})\big)$ with

$$
\begin{aligned}
g_{1\ell}(\mathbf{z}) &= \int_0^\phi A_{\ell+2}^{+}(s,\varphi(s,\mathbf{z}))\,ds+\int_\phi^{2\pi} A_{\ell+2}^{-}(s,\varphi(s,\mathbf{z}))\,ds,\\
g_{1\omega}(\mathbf{z}) &= \int_0^\phi e^{\mu_\omega(\phi-s)}\big(A_{\omega+2}^{+}(s,\varphi(s,\mathbf{z}))-\mu_\omega z_\omega A_1^{+}(s,\varphi(s,\mathbf{z}))\big)ds\\
&\quad+\int_\phi^{2\pi} e^{\mu_\omega(\phi-2\pi-s)}\big(A_{\omega+2}^{-}(s,\varphi(s,\mathbf{z}))-\ \mu_\omega z_\omega A_1^{-}(s,\varphi(s,\mathbf{z}))\big)ds,
\end{aligned}
\tag{19}
$$

for $0\le\ell\le m$ and $m+1\le\omega\le d$. Therefore the averaged function $f_1:\overline{V}\to\mathbb{R}^{m+1}$, defined in (6), is given by

$$
f_1(\nu)=\xi g_1(z_\nu)=\big(f_{10}(\nu),\ldots,f_{1m}(\nu)\big),
\tag{20}
$$

with $f_{1\ell}(\nu)=g_{1\ell}(\mathbf{z}_\nu)$, where $g_{1\ell}$ is given in (19) for $0\le\ell\le m$.

Now we compute the averaged function f_2 defined also in (6).

Since g_0 is linear (see (8) and (9)) we have $\dfrac{\partial^2\xi g_0}{\partial v^2}(\mathbf{z}_\nu)=0$. Moreover, as $\xi^\perp g_1(\mathbf{z}_\nu)=\big(g_{m+1}(\mathbf{z}_\nu),\ldots,g_d(\mathbf{z}_\nu)\big)$, it follows from (7) and (19) that $\gamma(\nu)=\big(\gamma_{m+1}(\nu),\ldots,\gamma_d(\nu)\big)$ where

$$
\gamma_\omega(\nu)=\frac{-1}{1-e^{-\mu_\omega 2\pi}}\left(\int_0^\phi e^{-s}A_{\omega+2}^{+}(s,\mathbf{z}_\nu)ds+\int_\phi^{2\pi} e^{-2\pi-s}A_{\omega+2}^{-}(s,\mathbf{z}_\nu)ds\right),
\tag{21}
$$

for $m+1\le\omega\le d$. Furthermore, for $v=(z_{m+1},\ldots,z_d)$ we have

$$
\frac{\partial\xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu)=\big(\widetilde{G}_{10}(\nu),\ldots,\widetilde{G}_{1m}(\nu)\big),
$$

with

$$\widetilde{G}_{1\ell}(\nu) = \sum_{\omega=m+1}^{d} \frac{\partial g_{1\ell}}{\partial z_\omega}(\mathbf{z}_\nu)\gamma_\omega(\nu), \tag{22}$$

where $g_{1\ell}$ is given in (19) for $0 \leq \ell \leq m$. Additionally from (8) and (9) we obtain

$$\xi g_2(\mathbf{z}_\nu) = \xi(y_2^+(\phi, \mathbf{z}_\nu)) - \xi(y_2^-(\phi - 2\pi, \mathbf{z}_\nu)),$$

where

$$\xi y_2^\pm(\theta, \mathbf{z}_\nu) = 2\int_0^\theta \xi\big(F_2^\pm(s, \mathbf{z}_\nu)\big) + \xi\left(\frac{\partial F_1^\pm}{\partial \mathbf{x}}(s, \mathbf{z}_\nu) y_1^\pm(s, \mathbf{z}_\nu)\right) ds,$$

because $F_0^\pm$ is linear.

On the other hand

$$\xi F_2^\pm(s, \mathbf{z}_\nu) = \big(F_{20}^\pm(s, \mathbf{z}_\nu), \ldots, F_{2m}^\pm(s, \mathbf{z}_\nu)\big), \text{ and}$$

$$\xi\left(\frac{\partial F_1^\pm}{\partial \mathbf{x}}(s, \mathbf{z}_\nu) y_1^\pm(s, \mathbf{z}_\nu)\right) = \big(\widetilde{F}_{10}^\pm(s, \mathbf{z}_\nu), \ldots, \widetilde{F}_{1m}^\pm(s, \mathbf{z}_\nu)\big),$$

being

$$\widetilde{F}_{1\ell}^\pm(s, \mathbf{z}_\nu) = \frac{\partial F_{1\ell}^\pm}{\partial r}(s, \mathbf{z}_\nu) y_{10}^\pm(s, \mathbf{z}_\nu) + \ldots + \frac{\partial F_{1\ell}^\pm}{\partial z_d}(s, \mathbf{z}_\nu) y_{1d}^\pm(s, \mathbf{z}_\nu), \tag{23}$$

for $F_{1\ell}^\pm$ and $F_{2\ell}^\pm$ defined in (17) for $0 \leq \ell \leq m$. Hence

$$f_2(\nu) = 2\frac{\partial \xi g_1}{\partial v}(\mathbf{z}_\nu)\gamma(\nu) + 2\,\xi g_2(\mathbf{z}_\nu) = \big(f_{20}(\nu), \ldots, f_{2m}(\nu)\big), \tag{24}$$

where

$$\begin{aligned} f_{2\ell}(\nu) = \;& 2\,\widetilde{G}_{1\ell}(\nu) + 2\int_0^\phi \big(F_{2\ell}^+(s, \mathbf{z}_\nu) + \widetilde{F}_{1\ell}^+(s, \mathbf{z}_\nu)\big) ds \\ & + 2\int_\phi^{2\pi} \big(F_{2\ell}^-(s, \mathbf{z}_\nu) + \widetilde{F}_{1\ell}^-(s, \mathbf{z}_\nu)\big) ds, \end{aligned} \tag{25}$$

for $0 \leq \ell \leq m$. See the explicit expression of all functions that appear in (25) in the Appendix.

If $m = d$, then the functions $\widetilde{G}_{1\ell}(\nu)$ are not considered because $f_2 = 2g_2$ (see Corollary 2).

3.3. **Some trigonometric integrals.** In order to study the zeros of the averaging functions f_1 and f_2, we need to know some results about trigonometric integrals. Then we shall state Lemma 8. The proof of this lemma will be omitted here, but it can easily be proven using some trigonometric relations found in Chapter 2 of [8].

For $p,q \in \mathbb{N}$ and $\phi \in (0,2\pi]$ consider the functions

$$I_{(p,q,\phi)} = \int_0^{\phi} \cos^p s \sin^q s \, ds, \quad J_{(p,q,\phi)} = \int_{\phi}^{2\pi} \cos^p s \sin^q s \, ds. \tag{26}$$

Lemma 8. *Let $I_{(p,q,\phi)}$ and $J_{(p,q,\phi)}$ be the functions defined in (26) for $\phi \in (0,2\pi]$. Then the following statements hold.*

(a) *If $\phi \neq \pi$ and $\phi \neq 2\pi$ then $I_{(p,q,\phi)}$, $J_{(p,q,\phi)}$, $\int_0^{\phi} \cos^i s \sin^j s \, I_{(p,q,\phi)} \, ds$, and $\int_{\phi}^{2\pi} \cos^i s \sin^j s \, I_{(p,q,\phi)} \, ds$ are non–zero;*

(b) *If $\phi = \pi$ then $I_{(p,q,\pi)} = 0$ or $J_{(p,q,\pi)} = 0$ if and only if p is odd. Moreover*

$$\int_0^{\pi} \cos^i s \sin^j s \, I_{(p,q,s)} \, ds = 0 \ \text{or} \ \int_{\pi}^{2\pi} \cos^i s \sin^j s \, I_{(p,q,s)} \, ds = 0$$

if and only if one of the following statements hold:
- (i) *i, j, p and q are odd;*
- (ii) *i, p and q are odd, and j is even;*
- (iii) *i and p are odd, and q and j are even;*
- (iv) *i, p and j are odd, and q is even.*

(c) *If $\phi = 2\pi$ then $I_{(p,q,2\pi)} \neq 0$ if and only if p and q are simultaneously even.*

4. Proof of Theorem 1

The proof of Theorem 1 is based on the next lemma which is a particular case of the *Lyapunov-Schmidt reduction* for a finite dimensional function (see for instance [4]).

Lemma 9. *Assuming $q \leq p$ are positive integers, let D and V be open bounded subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Let $g : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^p$ and $\sigma : \overline{V} \to \mathbb{R}^{p-q}$ be $\mathcal{C}^3$ functions such that $g(\mathbf{z},\varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^3)$ and $\mathcal{Z} = \{\mathbf{z}_\nu = (\nu, \sigma(\nu)) : \nu \in \overline{V}\} \subset D$. We denote by Γ_ν the upper right corner $q \times (p-q)$ matrix of $D\, g_0(\mathbf{z}_\nu)$, and by Δ_ν the lower right corner $(p-q) \times (p-q)$ matrix of $D\, g_0(\mathbf{z}_\nu)$. Assume that for each $\mathbf{z}_\nu \in \mathcal{Z}$, $\det(\Delta_\nu) \neq 0$ and $g_0(\mathbf{z}_\nu) = 0$.*

We consider the functions $f_1, f_2 : \overline{V} \to \mathbb{R}^q$ *defined in* (6). *Then the following statements hold.*

(a) *If there exists* $\nu^* \in V$ *with* $f_1(\nu^*) = 0$ *and* $\det(D\, f_1(\nu^*)) \neq 0$*, then there exists* ν_ε *such that* $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ *and* $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ *when* $\varepsilon \to 0$.

(b) *Assume that* $f_1 = 0$. *If there exists* $\nu^* \in V$ *with* $f_2(\nu^*) = 0$ *and* $\det(D\, f_2(\nu^*)) \neq 0$*, then there exists* ν_ε *such that* $g(\mathbf{z}_{\nu_\varepsilon}, \varepsilon) = 0$ *and* $\mathbf{z}_{\nu_\varepsilon} \to \mathbf{z}_{\nu^*}$ *when* $\varepsilon \to 0$.

The proof of this lemma can be found in [17].

Note that in Lemma 9 the functions g_i for $i = 0, 1, 2$ which appears in the expression of (6) and (7) are the ones of the function

$$g(z, \varepsilon) = g_0(z) + \varepsilon g_1(z) + \varepsilon^2 g_2(z) + \mathcal{O}(\varepsilon^3), \tag{27}$$

instead of the functions which appear in (8).

Proof of Theorem 1. Let $\psi(\theta, \mathbf{z}, \varepsilon)$ be a periodic solution of system (1) such that $\psi(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. Similarly let $\psi^{\pm}(\theta, \mathbf{z}, \varepsilon)$ be the solutions of the systems $\mathbf{x}' = F^{\pm}(\theta, \mathbf{x}, \varepsilon)$ such that $\psi^{\pm}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. So

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi \leq \theta \leq T. \end{cases}$$

Since the vector field (1) is T–periodic, it may also read

$$\psi(\theta, \mathbf{z}, \varepsilon) = \begin{cases} \psi^+(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad 0 \leq \theta \leq \phi, \\ \psi^-(\theta, \mathbf{z}, \varepsilon) & \text{if} \quad \phi - T \leq \theta \leq 0. \end{cases}$$

Now we consider the function $g(\mathbf{z}, \varepsilon) = \psi^+(\phi, \mathbf{z}, \varepsilon) - \psi^-(\phi - T, \mathbf{z}, \varepsilon)$. It is easy to see that the solution $\psi(\theta, \mathbf{z}, \varepsilon)$ is T–periodic in θ if and only if $g(\mathbf{z}, \varepsilon) = 0$. So from hypothesis (H) we have that $g(\mathbf{z}_\nu) = 0$ for every $\mathbf{z}_{\nu,\varepsilon} \in \mathcal{Z}$.

Using Taylor series to expand the functions $\psi^{\pm}(\theta, \mathbf{z}, \varepsilon)$ in powers of ε we obtain

$$\psi^{\pm}(\theta, \mathbf{z}, \varepsilon) = y_0^{\pm}(\theta, \mathbf{z}) + \varepsilon y_1^{\pm}(\theta, \mathbf{z}) + \varepsilon^2 \frac{y_2^{\pm}(\theta, \mathbf{z})}{2} + \mathcal{O}(\varepsilon^2), \tag{28}$$

where $y_i(\theta, \mathbf{z})$ is given in (9). We shall omit the computations for obtaining (28), nevertheless they can be found in [18]. Therefore $g(\mathbf{z}, \varepsilon) = g_0(\mathbf{z}) + \varepsilon g_1(\mathbf{z}) + \varepsilon^2 g_2(\mathbf{z}) + \mathcal{O}(\varepsilon^2)$, where $g_i(\mathbf{z}) = y_i^+(\phi, \mathbf{z}) - g_i^-(\phi - T, \mathbf{z})$ for $i = 0, 1, 2$. Moreover

$$Dg_0(\mathbf{z}) = \frac{\partial \varphi^+}{\partial \mathbf{z}}(\phi, \mathbf{z}) - \frac{\partial \varphi^-}{\partial \mathbf{z}}(\phi - T, \mathbf{z}) = Y^+(\phi, \mathbf{z}) - Y^-(\phi - T, \mathbf{z}).$$

So from hypothesis (H) we have that the matrix $Dg_0(\mathbf{z})$ has in the upper right corner the zero $q\times(d-q)$ matrix, and in the lower right corner has the $(p-q)\times(p-q)$ matrix Δ_ν with $\det(\Delta_\nu)\neq 0$.

The proof of this theorem concludes B

By applying Lemma 9 to the function $g(\mathbf{z},\varepsilon)$ defined in (27) it follows the proof of the theorem. □

5. Proof of Theorems 3 and 4

In order to prove Theorems 3 and 4 we shall study the zeros of the averaging functions f_1 and f_2, given in (20) and (24), respectively, when $\phi\in(0,2\pi)\setminus\{\pi\}$.

Remark 10. *For sake of simplicity we shall denote by $\lambda_{ijk_1\ldots k_m0}$ the coefficient of $x^iy^jz_1^{k_1}\ldots z_m^{k_m}$, and by λ_{ij0} the coefficient of x^iy^j of system (10), when $\lambda = a^\pm, b^\pm, c_\ell^\pm$ for all $1\leq \ell\leq m$.*

From statement (a) of Lemma 8 we have $f_1(\nu)=(f_{10}(\nu),\ldots,f_{1\ell}(\nu))$ where (29)

$$\begin{aligned} f_{10}(\nu) = & \sum_{i+j+k_1+\ldots+k_m=0}^{n} r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\Big(a^+_{ijk_1\ldots k_m0}I_{(i+1,j,\phi)} \\ & + b^+_{ijk_1\ldots k_m0}I_{(i,j+1,\phi)} + a^-_{ijk_1\ldots k_m0}J_{(i+1,j,\phi)} + b^-_{ijk_1\ldots k_m0}J_{(i,j+1,\phi)}\Big), \\ f_{1\ell}(\nu) = & \sum_{i+j+k_1+\ldots+k_m=0}^{n} r^{i+j}z_1^{k_1}\ldots z_m^{k_m}\Big(c^+_{\ell,ijk_1\ldots k_m0}I_{(i,j,\phi)} + c^-_{\ell,ijk_1\ldots k_m0}J_{(i,j,\phi)}\Big), \end{aligned}$$

here $\nu=(r,z_1,\ldots,z_m)$ and $1\leq\ell\leq m$.

Proposition 11. *Assume $0\leq m\leq d$ and $\phi\neq\pi$. Then f_1 has at most n^{m+1} simple zeros and this number can be reached.*

Proof. For each $0\leq\ell\leq m$ and $\nu=(r,z_1,\ldots,z_m)$, $f_{1\ell}(\nu)$ is a complete polynomial of degree n. Recall that a *complete polynomial of degree k* means a polynomial that appears all its monomials. By Bezout Theorem (see [7]), $f_1(\nu)$ can be at most n^{m+1} simple zeros. Since all the coefficients of $f_1(\nu)$ are independent, we can choose them in order that $f_1(\nu)$ has exactly n^{m+1} zeros with $r>0$, and $\det f_1'(\nu^*)\neq 0$ for each zero $r\nu^*$ of $f_1(\nu)$ (that is, ν^* is a simple zero). □

Proposition 12. *Take $0\leq m\leq d$ and $\phi\neq\pi$. If $f_1\equiv 0$ then f_2 has at most $(2n)^{m+1}$ simple zeros, and a lower bound for the maximum number of simple zeros is $(2n)(2n-1)^m$.*

Proof. Assume that $f_1 \equiv 0$. From (29) it follows that

$$\begin{aligned}
&\sum_{i+j=s} a^+_{ijk_1\ldots k_m0}\, I_{(i+1,j,\phi)} + b^+_{ijk_1\ldots k_m0}\, I_{(i,j+1,\phi)} \\
&\qquad + a^-_{ijk_1\ldots k_m0}\, J_{(i+1,j,\phi)} + b^-_{ijk_1\ldots k_m0}\, J_{(i,j+1,\phi)} = 0, \\
&\sum_{i+j=s} c^+_{\ell,ijk_1\ldots k_m0}\, I_{(i,j,\phi)} + \, c^-_{\ell,ijk_1\ldots k_m0}\, J_{(i,j,\phi)} = 0,
\end{aligned} \tag{30}$$

for $1 \le \ell \le m$, $0 \le s \le n$, $0 \le k_\ell \le n$ with $0 \le k_1 + \ldots + k_m \le n - s$.

Moreover $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$ with $\nu = (r, z_1, \ldots, z_m)$. In particular, if $m = 0$ then $f_2(\nu) = f_{20}(r)$. Considering the expression for $f_{2\ell}(\nu)$, given in (25) for $0 \le \ell \le m$, we conclude that $\widetilde{G}_{1\ell}(\nu)$ and $\displaystyle\int_0^\phi \widetilde{F}^+_{1\ell}(s, \mathbf{z}_\nu)\, ds + \int_\phi^{2\pi} \widetilde{F}^-_{1\ell}(s, \mathbf{z}_\nu)\, ds$ are complete polynomials of degree $2n - 1$ in the variables $(r, z_1, \ldots, z_m)$, and

$$\int_0^\phi F^+_{2\ell}(s, \mathbf{z}_\nu)\, ds + \int_\phi^{2\pi} F^-_{2\ell}(s, \mathbf{z}_\nu)\, ds = \frac{1}{r} \sum_{k=0}^{2n} Q_k(z_1, \ldots, z_m)\, r^k,$$

where $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0) \in \mathbb{R}^{d+1}$, $Q_k(z_1, \ldots, z_m)$ is a complete polynomial of degree $2n - k$ in the variables $(z_1, \ldots, z_m)$ if $m \neq 0$, and $Q_k(z_1, \ldots, z_m)$ is constant if $m = 0$. The above equality is evident if we take into account statement (a) of Lemma 8 and conditions (30). Therefore, each $r f_{2\ell}(\nu)$ is a complete polynomial of degree $2n$ in the variables $(r, z_1, \ldots, z_m)$. Since $r > 0$, it is known that $r f_{2\ell}(\nu) = 0$ if and only if $f_{2\ell}(\nu) = 0$ for each $0 \le \ell \le m$. Then by Bezout Theorem, $f_2(\nu)$ has at most $(2n)^{m+1}$ simple zeros for all $0 \le m \le d$.

In order to show that the maximum number is greater than or equal to $(2n)(2n-1)^m$ we provide a particular example. So take $a^\pm_{i00} \neq 0$, $c^\pm_{\ell,00\ldots0k_\ell0} \neq 0$, and we take zero all the other coefficients for $1 \le \ell \le m$. From (25) we obtain $f_{20}(\nu) = f_{20}(r)$ and $f_{2\ell}(\nu) = f_{2\ell}(r, z_\ell)$, where

$$\begin{aligned}
f_{20}(r) = \; & \frac{2}{r} \sum_{i=0}^{n} \sum_{p=0}^{n} r^{i+p} \big(a^+_{i00} a^+_{p00} I_{(i+p+1,1,\phi)} + a^-_{i00} a^-_{p00} J_{(i+p+1,1,\phi)} \\
& + i\, a^+_{i00} a^+_{p00} \int_0^\phi \cos^{i+1} s\, I_{(p+1,0,s)} ds + \, i\, a^-_{i00} a^-_{p00} \int_\phi^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds \big),
\end{aligned}$$

$$
\begin{aligned}
f_{2\ell}(r,z_\ell)= \;& \frac{2}{r}\sum_{i=0}^{n}\sum_{k_l=0}^{n} r^i z_\ell^{k_\ell}\big(a_{i00}^{+}c_{\ell,0\dots0k_\ell0}^{+}I_{(i,1,\phi)}+a_{i00}^{-}c_{\ell,0\dots0k_\ell0}^{-}J_{(i,1,\phi)}\big)\\
&+\sum_{k_\ell=1}^{n}\sum_{L_\ell=0}^{n} z_\ell^{k_\ell+L_\ell-1}\Bigg(\tfrac{\phi^2}{2}k_\ell\, c_{\ell,0\dots0k_\ell0}^{+}\, c_{\ell,0\dots0L_\ell0}^{+}\\
&+\tfrac{(2\pi)^2-\phi^2}{2}k_\ell\, c_{\ell,0\dots0k_\ell0}^{-}\, c_{\ell,0\dots0L_\ell0}^{-}\Bigg),
\end{aligned}
$$

where $a_{i00}^{+}I_{(i+1,0,\phi)}=-a_{i00}^{-}J_{(i+1,0,\phi)}$ and $c_{\ell,00\dots0k_\ell0}^{+}I_{(0,0,\phi)}=-c_{\ell,00\dots0k_\ell0}^{-}J_{(0,0,\phi)}$ for $1\le\ell\le m$ (see (30)).

From statement (a) of Lemma 8, $rf_{20}(r)$ is a complete polynomial of degree $2n$ in the variable r, whose coefficients are independent. Furthermore, if $f_{20}(r^*)=0$ with $r^*>0$, then $f_{2\ell}(r^*,z_\ell)$ is a polynomial of degree $2n-1$ in the variable z_ℓ, and all their coefficients are independent for $1\le\ell\le m$. Therefore By Bezout Theorem, $f_2(\nu)$ has at most $(2n)(2n-1)^m$ simple zeros, and this number can be reached due to the independence of coefficients. □

Corollary 13. *If $m=1$ in Proposition 12 then f_2 has at most $(2n)^2$ simple zeros and this number can be reached.*

Proof. If $m=1$ we have $f_2(\nu)=(f_{20}(\nu),f_{21}(\nu))$ with $\nu=(r,z_1,\dots,z_m)$. From Proposition 12 we know that $f_2(\nu)$ has at most $(2n)^2$ simple zeros. Now we provide a particular example to prove that this number is reached. Take $a_{i00}^{\pm}\neq0$, $a_{00k_10}^{\pm}\neq0$, $c_{1,00k_10}^{\pm}\neq0$ and we take zero all the other coefficients, such that $f_{20}(\nu)=f_{20}(r)$ being

$$
f_{20}(r)=\frac{2}{r}\left(\sum_{i=0}^{n}\sum_{p=0}^{n} r^{i+p}d_{ip}+\sum_{k_1=0}^{n}\sum_{L_1=0}^{n} z_1^{k_1+L_1}D_{k_1L_1}\right)+\sum_{k_1=0}^{n}\sum_{L_1=0}^{n} z_1^{k_1+L_1-1}E_{k_1L_1},
$$

where

$$
\begin{aligned}
d_{ip} &= a_{i00}^{+}a_{p00}^{+}I_{(i+p+1,1,\phi)}+a_{i00}^{-}a_{p00}^{-}J_{(i+p+1,1,\phi)}\\
&\quad + i\,a_{i00}^{+}a_{p00}^{+}\int_0^{\phi}\cos^{i+1}s\,I_{(p+1,0,s)}\,ds+\,i\,a_{i00}^{-}a_{p00}^{-}\int_{\phi}^{2\pi}\cos^{i+1}s\,I_{(p+1,0,s)}\,ds,\\
D_{k_1L_1} &= a_{00k_10}^{+}a_{00L_10}^{+}I_{(1,1,\phi)}+a_{00k_10}^{-}a_{00L_10}^{-}J_{(1,1,\phi)},\\
E_{k_1L_1} &= k_1a_{00k_10}^{+}c_{1,00L_10}^{+}\int_0^{\phi}\frac{s^2}{2}\cos s ds+\,k_1a_{00k_10}^{-}c_{1,00L_10}^{-}\int_{\phi}^{2\pi}\frac{s^2}{2}\cos s\,ds,
\end{aligned}
$$

for $a^+_{i00}I_{(i+1,0,\phi)} = -a^-_{i00}J_{(i+1,0,\phi)}$, $a^+_{00k_10}I_{(1,0,\phi)} = -a^-_{00k_10}J_{(1,0,\phi)}$ and $c^+_{1,00k_10}I_{(0,0,\phi)} = -c^-_{1,00k_10}J_{(0,0,\phi)}$. Therefore we can take $D_{k_1L_1} = E_{k_1L_1} = 0$ because all the coefficients of $f_{20}(r)$ are independent. In a similar way we have $f_{21}(\nu) = f_{21}(r, z_1)$ where

$$\begin{aligned} f_{21}(r,z_1) = \; & \frac{2}{r}\sum_{i=0}^{n}\sum_{k_1=0}^{n} r^i z_1^{k_1}\left(a^+_{i00}c^+_{1,00k_10}I_{(i,1,\phi)} + a^-_{i00}c^-_{1,00k_10}J_{(i,1,\phi)}\right) \\ & + \sum_{k_1=1}^{n}\sum_{L_1=0}^{n} z_1^{k_1+L_1-1}k_1\left(\frac{\phi^2}{2}\, c^+_{1,00k_10}\, c^+_{1,00L_10} + \frac{(2\pi)^2-\phi^2}{2}\, c^-_{1,00k_10}\, c^-_{1,00L_10}\right), \\ & + \frac{1}{r}\sum_{k_1=0}^{n}\sum_{L_1=0}^{n} z_1^{k_1+L_1}\left(a^+_{00k_10}c^+_{1,00k_10\ldots0}I_{(i,1,\phi)} + a^-_{00k_10}c^-_{1,00L_10\ldots0}J_{(i,1,\phi)}\right). \end{aligned}$$

Therefore $rf_{20}(r)$ is a complete polynomial of degree $2n$ in the variable r, and if $f_{20}(r^*) = 0$ with $r^* > 0$, then $r^*f_{21}(r^*, z_1)$ is a complete polynomial of degree $2n$ in the variable z_1. Since $c^\pm_{1,00k_10}$ depends just of $a^\pm_{00k_10}$, which do not appear in the expression of $f_{20}(r)$, then all coefficients of $f_2(\nu)$ are independent. So, they can be chosen in order to $f_2(\nu)$ has $(2n)^2$ simple zeros with $r > 0$. □

Proof of Theorem 3. We apply Theorem 1 to the function f_1 of Proposition 11 and we conclude statement (a). Statement (b) is proved applying Theorem 1 to the functions f_1 and f_2 given in Proposition 12. Finally, applying Theorem 1 to the functions f_1 and f_2 in Corollary 13, statement (c) holds. □

The proof of Theorem 4 follows from the next Lemma.

Lemma 14. *For $0 < m \le d$ the lower bound of the maximum number of limit cycles of the discontinuous system* (10) *is controlled by the number of solutions of the polynomial equation system*

$$\begin{aligned} & rP_0(\nu) + Q_0(\nu) = 0, \\ & \vdots \\ & rP_m(\nu) + Q_m(\nu) = 0, \end{aligned}$$

for $\nu = (r, z_1, \ldots, z_m)$. The polynomials $P_\ell(\nu)$ and $Q_\ell(\nu)$ for $\ell = 0, 1, \ldots, m$ are defined in the proof of this lemma.

Proof. From Theorem 1 we need to know the zeros of $f_2(\nu)$ when $f_1 \equiv 0$. From (24) we have $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$. In (25) we can take $\widetilde{G}_{1\ell}(\nu) = 0$. Since $r > 0$, we know that $f_2(\nu^*) = 0$ if and only if $\dfrac{rf_2(\nu^*)}{\delta} = 0$.

Defines $\widetilde{A}_1^{\pm} = rA_1^{\pm}$, $\widetilde{A}_{\ell+2}^{\pm} = \dfrac{A_{\ell+2}^{\pm}}{\delta}$ and $\widetilde{B}_{\ell+2}^{\pm} = \dfrac{B_{\ell+2}^{\pm}}{\delta}$ for $\delta > 0$ sufficiently small, and take

$$
\begin{aligned}
P_\ell(\nu) = &\int_0^{\phi} \widetilde{B}_{\ell+2}^{+}(s, \mathbf{z}_\nu)\, ds + \int_{\phi}^{2\pi} \widetilde{B}_{\ell+2}^{-}(s, \mathbf{z}_\nu)\, ds, \\
Q_\ell(\nu) = &\int_0^{\phi} \widetilde{A}_1^{+}(s, \mathbf{z}_\nu)\widetilde{A}_{\ell+2}^{+}(s, \mathbf{z}_\nu)\, ds + \int_{\phi}^{2\pi} \widetilde{A}_1^{-}(s, \mathbf{z}_\nu)\widetilde{A}_{\ell+2}^{-}(s, \mathbf{z}_\nu)\, ds,
\end{aligned}
$$

for $0 \le \ell \le m$. From (17) and (18) we have $\displaystyle\int \widetilde{F}_{1\ell}^{\pm}(s, \mathbf{z}_\nu) ds = \mathcal{O}_2(\delta)$. Therefore

$$
\frac{r f_{2\ell}(\nu)}{2\delta} = \quad rP_\ell(\nu) + Q_\ell(\nu) + \mathcal{O}(\delta), \quad \text{for } 0 \le \ell \le m.
$$

Since δ is sufficiently small the lemma is proved. □

6. Proof of Theorem 6

In this section we study the zeros of the functions f_1 and f_2, given in (20) and (24), respectively, when $\phi = \pi$. Then we conclude Theorem 6 applying Theorem 1.

From statement (b) of Lemma 8 we have $f_1(\nu) = (f_{10}(\nu), \dots, f_{1\ell}(\nu))$ where (31)

$$
\begin{aligned}
f_{10}(\nu) = &\sum_{i\,odd,\, P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left(a_{ijk_1\dots k_m 0}^{+} I_{(i+1,j,\pi)} + a_{ijk_1\dots k_m 0}^{-} J_{(i+1,j,\pi)} \right) \\
&+ \sum_{i\,even,\, P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left(b_{ijk_1\dots k_m 0}^{+} I_{(i,j+1,\pi)} + b_{ijk_1\dots k_m 0}^{-} J_{(i,j+1,\pi)} \right), \\
f_{1\ell}(\nu) = &\sum_{i\,even,\, P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \left(c_{\ell,ijk_1\dots k_m 0}^{+} I_{(i,j,\pi)} + c_{\ell,ijk_1\dots k_m 0}^{-} J_{(i,j,\pi)} \right),
\end{aligned}
$$

where $\nu = (r, z_1, \dots, z_m)$, $1 \le \ell \le m$ and $P = i + j + k_1 + \dots + k_m$.

Proposition 15. *Take $0 \le m \le d$ and $\phi = \pi$. Then f_1 has at most n^{m+1} simple zeros and this number can be reached.*

Proof. This proof is analogously to the proof of Proposition 11, noticing that for each $0 \le \ell \le m$, $f_{1\ell}(\nu)$ is a complete polynomial of degree n in the variables $(r, z_1, \dots, z_m)$ and all their coefficients are independent. □

Proposition 16. *Assume $0 \le m \le d$ and $\phi = \pi$. If $f_1 \equiv 0$ then f_2 has at most $(2n)^{m+1}$ simple zeros, and the lower bound for the number of simple zeros is $(2n-1)^{m+1}$ if n is odd, and $(2n-2)(2n-1)^m$ if n is even.*

Proof. Assume that $f_1 \equiv 0$. From (31) it follows that

$$
\begin{aligned}
&\sum_{i\,odd,\, i+j=s} a^{+}_{ijk_1\ldots k_m 0}\, I_{(i+1,j,\pi)} + a^{-}_{ijk_1\ldots k_m 0}\, J_{(i+1,j,\pi)} \\
&+ \sum_{i\,even,\, i+j=s} b^{+}_{ijk_1\ldots k_m 0}\, I_{(i,j+1,\pi)} + b^{-}_{ijk_1\ldots k_m 0}\, J_{(i,j+1,\pi)} = 0, \\
&\sum_{i\,even,\, i+j=s} c^{+}_{\ell,ijk_1\ldots k_m 0}\, I_{(i,j,\pi)} + c^{-}_{\ell,ijk_1\ldots k_m 0}\, J_{(i,j,\pi)} = 0,
\end{aligned} \tag{32}
$$

for $1 \le \ell \le m,\ 0 \le s \le n,\ 0 \le k_\ell \le n$ with $0 \le k_1 + \ldots + k_m \le n - s$.

Moreover $f_2(\nu) = (f_{20}(\nu), \ldots, f_{2m}(\nu))$ with $\nu = (r, z_1, \ldots, z_m)$. If $m = 0$ then $f_2(\nu) = f_{20}(r)$. Analogously to the proof of Proposition 12 we conclude that $f_2(\nu)$ has at most $(2n)^{m+1}$ simple zeros for all $0 \le m \le d$.

Now we provide a particular example to exhibit the lower bound for the maximum number of simple zeros. So take $a^{\pm}_{i00} \neq 0$, $c^{\pm}_{\ell,00\ldots0k_\ell 0} \neq 0$, and take zero all the other coefficients for $1 \le \ell \le m$. From (25) we obtain $f_{20}(\nu) = f_{20}(r)$ and $f_{2\ell}(\nu) = f_{2\ell}(r, z_\ell)$, where

$$
\begin{aligned}
f_{20}(r) = \; & \frac{2}{r}\Bigg(\sum_{i\,even,\, i=0}^{n} \; \sum_{p\,odd,\, p=0}^{n} r^{i+p}\big(a^{+}_{i00}a^{+}_{p00} I_{(i+p+1,1,\pi)} + a^{-}_{i00}a^{-}_{p00} J_{(i+p+1,1,\pi)} \\
& + i\, a^{+}_{i00}a^{+}_{p00} \int_0^{\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds + i\, a^{-}_{i00}a^{-}_{p00} \int_{\pi}^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds\big) \\
& + \sum_{i\,odd,\, i=0}^{n} \; \sum_{p\,even,\, p=0}^{n} r^{i+p}\big(a^{+}_{i00}a^{+}_{p00} I_{(i+p+1,1,\pi)} + a^{-}_{i00}a^{-}_{p00} J_{(i+p+1,1,\pi)} \\
& + i\, a^{+}_{i00}a^{+}_{p00} \int_0^{\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds + i\, a^{-}_{i00}a^{-}_{p00} \int_{\pi}^{2\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds\big) \\
& + \sum_{i\,odd,\, i=0}^{n} \; \sum_{p\,odd,\, p=0}^{n} r^{i+p}\big(a^{+}_{i00}a^{+}_{p00} I_{(i+p+1,1,\pi)} + a^{-}_{i00}a^{-}_{p00} J_{(i+p+1,1,\pi)} \\
& + i\, a^{+}_{i00}a^{+}_{p00} \int_0^{\pi} \cos^{i+1} s\, I_{(p+1,0,s)} ds + i a^{-}_{i00}a^{-}_{p00} \int_{\pi}^{2\pi} \cos^{i+1} s I_{(p+1,0,s)} ds\big)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
f_{2\ell}(r, z_\ell) = \; & \frac{2}{r} \sum_{i=0}^{n} \sum_{k_l=0}^{n} r^i z_\ell^{k_\ell} \big(a^{+}_{i00} c^{+}_{\ell,0\ldots0k_\ell 0} I_{(i,1,\phi)} + a^{-}_{i00} c^{-}_{\ell,0\ldots0k_\ell 0} J_{(i,1,\phi)}\big) \\
& + \sum_{k_\ell=1}^{n} \sum_{L_\ell=0}^{n} z_\ell^{k_\ell+L_\ell-1} k_\ell \left(\tfrac{\phi^2}{2} c^{+}_{\ell,0\ldots0k_\ell 0} c^{+}_{\ell,0\ldots0L_\ell 0} + \tfrac{(2\pi)^2-\phi^2}{2} c^{-}_{\ell,0\ldots0k_\ell 0} c^{-}_{\ell,0\ldots0L_\ell 0} \right),
\end{aligned}
$$

for $1 \leq \ell \leq m$, where $a^+_{i00} I_{(i+1,0,\pi)} = -a^-_{i00} J_{(i+1,0,\pi)}$ if i is odd and $c^+_{\ell,00\dots0k_\ell 0} I_{(0,0,\pi)} = -c^-_{\ell,00\dots0k_\ell 0} J_{(0,0,\pi)}$ (see (32)). Therefore from statement (b) of Lemma 8, $r f_{20}(r)$ is a complete polynomial in the variable r of degree $2n-1$ if n is odd, and $2n-2$ if n is even, and its coefficients are independent. Furthermore, if $f_{20}(r^*) = 0$ with $r^* > 0$, then $f_{2\ell}(r^*, z_\ell)$ is a polynomial of degree $2n-1$ in the variable z_ℓ for each $1 \leq \ell \leq m$. Then the number of simple zeros with $r > 0$ of $f_2(\nu)$ can be $(2n-1)^{m+1}$ if n is odd, and $(2n-2)(2n-1)^m$ if n is even. By the independence of all coefficients these numbers can be reached. □

Proof of Theorem 6. From Theorem 1 and Proposition 15, statement (a) holds, and applying Theorem 1 to the functions f_1 and f_2 given in Proposition 16 we conclude statement (b). □

7. Proof of Theorem 7

When $\phi = 2\pi$ system (10) is continuous. Then considering the cylindrical coordinates given in (12), and taking θ as the new time, system (10) can be written as system (1) that is,

$$\mathbf{x}' = F^+(\theta, \mathbf{x}, \varepsilon), \quad \text{for } 0 \leq \theta \leq 2\pi,$$

where

$$F^+(\theta, \mathbf{x}, \varepsilon) = F_0^+(\theta, \mathbf{x}) + \varepsilon F_1^+(\theta, \mathbf{x}) + \varepsilon^2 F_2^+(\theta, \mathbf{x}) + \varepsilon^3 R^+(\theta, \mathbf{x}, \varepsilon),$$

for $\mathbf{x} = (r, \mathrm{z})$ and $\mathrm{z} = (z_1, \dots, z_d)$, with F_j^+ given in (16) and (17) for $j = 0, 1, 2$.

From statement (c) of Lemma 8 we have $f_1(\nu) = (f_{10}(\nu), \dots, f_{1m}(\nu)$ with

$$\begin{aligned} f_{10}(\nu) &= \sum_{i\,odd,\, j\,even,\, P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m}\, a^+_{ijk_1\dots k_m 0} I_{(i+1,j,2\pi)} \\ &\quad + \sum_{i\,even,\, j\,odd,\, P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m}\, b^+_{ijk_1\dots k_m 0} I_{(i,j+1,2\pi)}, \\ f_{1\ell}(\nu) &= \sum_{i,j\,even,\, P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m}\, c^+_{\ell,ijk_1\dots k_m 0} I_{(i,j,2\pi)}, \end{aligned} \tag{33}$$

where $\nu = (r, z_1, \dots, z_m)$, $1 \leq \ell \leq m$ and $P = i + j + k_1 + \dots + k_m$.

Proposition 17. *Assume $0 \leq m \leq d$ and $\phi = 2\pi$. If $m \neq 0$ then f_1 has at most n^{m+1} simple zeros and this number can be reached. If $m = 0$ then f_1 has at most $(n-1)/2$ simple zeros if n is odd, and $(n-2)/2$ if n is even, and these numbers can be reached.*

Proof. We have $f_{10}(\nu) = r\widetilde{f}_{10}(\nu)$ with

$$\widetilde{f}_{10}(\nu) = h_1 + r^2h_3 + r^4h_5 + r^6h7 + \ldots + \begin{cases} r^{n-1}h_n & \text{if n is odd,} \\ r^{n-2}h_{n-1} & \text{if n is even,} \end{cases}$$

where

$$\begin{aligned} h_k = & \sum_{k_1+\ldots+k_m=0}^{n-k} z_1^{k_1}\ldots z_m^{k_m}\Bigg(\sum_{i\,odd,\,j\,even,\,i+j=k} a^+_{ijk_1\ldots k_m0}\, I_{(i+1,j,2\pi)} \\ & + \sum_{i\,even,\,j\,odd,\,i+j=k} b^+_{ijk_1\ldots k_m0}\, I_{(i,j+1,2\pi)}\Bigg). \end{aligned}$$

If $m \neq 0$ then $\widetilde{f}_{10}(\nu)$ and $f_{1\ell}(\nu)$ are polynomials in the variables $(r, z_1, \ldots, z_m)$ of degree $n-1$ and n, respectively, for $1 \leq \ell \leq m$. From Bezout Theorem the maximum number of simple zeros of $f_1(\nu)$ is $n^m(n-1)$. Since the exponents of r in the function $\widetilde{f}_{10}(\nu)$ are always even numbers, the maximum number of simple zeros of $f_1(\nu)$ is $n^m(n-1)/2$. In what follows we provide a particular example to prove that this number is reached.

First if n is even we take $a^+_{10k_10} \neq 0$, $b^+_{01k_10} \neq 0$, $c^+_{1,ij0} \neq 0$, $c^+_{\ell,00k_\ell 0} \neq 0$ and take zero all the other coefficients in other that $\widetilde{f}_{10}(\nu) = \widetilde{f}_{10}(z_1)$, $f_{11}(\nu) = f_{11}(r)$, and $f_{1\ell}(\nu) = f_{11}(z_\ell)$, where

$$\begin{aligned} \widetilde{f}_{10}(z_1) &= \sum_{k_1=0}^{n-1} z_1^{k_1}\left(a^+_{10k_10}I_{(2,0,2\pi)} + b^+_{01k_10}I_{(0,2,2\pi)}\right), \\ f_{11}(r) &= \sum_{i,j\,even,\,i+j=0}^{n} r^{i+j}\, c^+_{1,ij0}\, I_{(i,j,2\pi)}, \\ f_{1\ell}(z_\ell) &= \sum_{k_\ell=0}^{n} z_\ell^{k_\ell}\, c^+_{\ell,00k_\ell 0}\, I_{(0,0,2\pi)}, \end{aligned}$$

for $2 \leq \ell \leq m$. Thus $\widetilde{f}_{10}(z_1)$ is a complete polynomial of degree $n-1$ in the variable z_1, $f_{11}(r)$ is an even polynomial of degree n in the variable r, and $f_{1\ell}(z_\ell)$ is a complete polynomial of degree n in the variable z_ℓ for all $2 \leq \ell \leq m$. Since the exponents of r in $f_{11}(r)$ is even, then $f_1(\nu)$ can have $n^m(n-1)/2$ simple zeros with $r > 0$.

On the other hand, if n is odd we take $a^+_{ij0} \neq 0$, $b^+_{ij0} \neq 0$, $c^+_{\ell,00k_\ell 0} \neq 0$ and we take zero all the other coefficients and then we obtain $\widetilde{f}_{10}(\nu) = \widetilde{f}_{10}(r)$ and

$f_{1\ell}(\nu) = f_{1\ell}(z_\ell)$, where

$$\begin{aligned} \widetilde{f}_{10}(r) =& \quad h_1 + rh_2 + r^2h_3 + \ldots + r^{n-1}h_n, \\ f_{1\ell}(\nu) =& \quad \sum_{k_\ell=0}^{n} {z_\ell}^{k_\ell}\, c_{\ell,00k_\ell 0}\, I_{(0,0,2\pi)}, \end{aligned}$$

for $2 \leq \ell \leq m$. Then $\widetilde{f}_{10}(r)$ is a polynomial of degree $n-1$ in the variable r, whose exponents are always even. In a similar way $f_{1\ell}(z_\ell)$ is a polynomial of degree n in the variable z_ℓ for $1 \leq \ell \leq m$. Therefore $f_1(\nu)$ can have $(n-1)n^m/2$ simple zeros with $r > 0$.

If $m = 0$ then $\nu = r$ and $f_1(\nu) = r\widetilde{f}_{10}(r)$. So the number of simple zeros can be $n-1$ if n is odd, and $n-2$ if n is even. Since the exponent of r in $\widetilde{f}_{10}$ is even, the maximum number of simple zeros with $r > 0$ of $f_1(\nu)$ is $(n-1)/2$ if n is odd, and $(n-2)/2$ if n is even.

Now we exhibit a particular example where the maximum number of simple zeros of $f_1(\nu)$ can be reached. Take $a^+_{ij0} \neq 0$, $b^+_{ij0} \neq 0$ and we take zero all the other coefficients so that $\widetilde{f}_{10}(r)$ is an even polynomial in the variable r of degree $n-1$ if n is odd, and $n-2$ is n if even. So the number of simple zeros of $f_1(\nu)$ with $r > 0$ can be $(n-1)/2$ if n is odd, and $(n-2)/2$ if n is even.

In both particular cases, $m \neq 0$ and $m = 0$, the coefficients of $f_1(\nu)$ are independent. Therefore the maximum number of simple zeros with $r > 0$ of $f_1(\nu)$ can be reached. □

Now we emphasize that the averaging function f_2 of the continuous system (10), for $\phi = 2\pi$, is given by $f_2(\nu) = \big(f_{20}(\nu), \ldots, f_{2m}(\nu)\big)$ being

$$f_{2\ell}(\nu) = \quad 2\widetilde{G}_{1\ell}(\nu) + 2\int_0^{2\pi} \big(F^+_{2\ell}(s, \mathbf{z}_\nu) + \widetilde{F}^+_{1\ell}(s, \mathbf{z}_\nu)\big)\, ds, \tag{34}$$

for $0 \leq \ell \leq m$, $F^+_{2\ell}$, $\widetilde{G}_{1\ell}$ and $\widetilde{F}^+_{1\ell}$ given in (17), (22) and (23), respectively.

Proposition 18. *Assume $m = 0$ and $\phi = 2\pi$. If $f_1 \equiv 0$ then f_2 has at most $2n$ simple zeros, and the lower bound for the number of simple zeros is n if n is even, and $n-1$ if n is odd.*

Proof. If $m = 0$ then $\nu = r$ and $f_1(\nu) = f_{10}(r)$. Assume that $f_1 \equiv 0$. From (33) we obtain

$$\sum_{i\, odd,\, j\, even,\, P=s}^{n} a^+_{ij0} I_{(i+1,j,2\pi)} + \sum_{i\, even,\, j\, odd,\, P=s}^{n} b^+_{ij0} I_{(i,j+1,2\pi)} = 0, \tag{35}$$

where $P = i + j$ and $0 \leq s \leq n$.

Furthermore by (34) we have $f_2(\nu) = f_{20}(r)$. Therefore, from statement (c) of Lemma 8 and (35), we conclude that $\widetilde{G}_{10}(\nu)$ and $\displaystyle\int_0^{2\pi} \widetilde{F}_{10}(s, \mathbf{z}_\nu)\, ds$ are complete polynomials of degree $2n-1$ in the variable r, and

$$\int_0^{2\pi} F_{20}^{+}(s, \mathbf{z}_\nu)\, ds = \sum_{s=0}^{N_1} R_s\, r^{2s+1} + \frac{1}{r}\sum_{k=0}^{N_2} Q_k\, r^{2k},$$

where $\mathbf{z}_\nu = (r, 0, \dots, 0) \in \mathbb{R}^{d+1}$, R_s and Q_k are constants, and $N_1 = \dfrac{n-2}{2}$ and $N_2 = n$ if n is even, and $N_1 = \dfrac{n-1}{2}$ and $N_2 = n-1$ if n is odd. Therefore, $r\displaystyle\int_0^{2\pi} F_{20}^{+}(s, \mathbf{z}_\nu)\, ds$ is an even polynomial in the variable r. Since $r > 0$ it follows that $rf_2(\nu) = 0$ if and only if $f_2(\nu) = 0$. By Bezout Theorem the maximum number of simple zeros of $f_2(\nu)$ is $2n$.

In order to exhibit the lower bound for the number of simple zeros of $f_2(\nu)$, we provide a particular example. Then take $a_{ij0}^{\pm} \neq 0$, $b_{ij0}^{\pm} \neq 0$, $\alpha_{ij0}^{\pm} \neq 0$, and $\beta_{ij0}^{\pm} \neq 0$ and we take zero all the other coefficients in such a way that $f_{20}(r) = 2\displaystyle\int_0^{2\pi} F_{20}^{+}(r, \mathbf{z}_\nu) d\theta$. Therefore $rf_{20}(r)$ is a polynomial in r of degree $2n$ if n is even, and $2n-2$ if n is odd. Since $rf_{20}(r)$ is an even polynomial in r, then the number of simple zeros of $f_2(\nu)$ with $r > 0$ can be n if n is even, and $n-1$ if n is odd, and these numbers can be reached due to the independence of all coefficients. □

Proof of Theorem 7. Applying Theorem 1 to the function f_1 given in Proposition 17, statement (a) holds. We apply Theorem 1 to the function f_2 given in Proposition 18 and we conclude statement (b). □

Appendix

In this appendix we shall exhibit some general expression of functions that appears in subsection 3.2.

Take $P = i + j + k_1 + \ldots + k_m$ and $Q = p + q + L_1 + \ldots + L_m$. We will denote by $\lambda_{ijk_1\ldots k_m 1_\omega}$ the coefficient of $x^i y^j z_1^{k_1} \ldots z_m^{k_m} z_\omega$, and by $\lambda_{ijk_1\ldots k_m 0}$ the coefficient of $x^i y^j z_1^{k_1} \ldots z_m^{k_m}$ in system (10), when $\lambda = a^{\pm}, b^{\pm}, \alpha^{\pm}, \beta^{\pm}, c_\ell^{\pm}, \gamma_\ell^{\pm}$ for all $0 \leq \ell \leq m$ and $m+1 \leq \omega \leq d$. Recall that $\nu = (r, z_1, \ldots, z_m)$ and $\mathbf{z}_\nu = (r, z_1, \ldots, z_m, 0, \ldots, 0) \in \mathbb{R}^{d+1}$.

From (14) and (19) we obtain

$$\begin{aligned}
\frac{\partial g_{10}}{\partial z_\omega}(\mathbf{z}_\nu) = & \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \Bigg(a^+_{ijk_1\dots k_m 1_\omega} \int_0^\phi e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds \\
& + b^+_{ijk_1\dots k_m 1_\omega} \int_0^\phi e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds \\
& + a^-_{ijk_1\dots k_m 1_\omega} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds \\
& + b^-_{ijk_1\dots k_m 1_\omega} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds \Bigg), \\
\frac{\partial g_{1\ell}}{\partial z_\omega}(\mathbf{z}_\nu) = & \sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \Bigg(c^+_{l,ijk_1\dots k_m 1_\omega} \int_0^\phi e^{\mu_\omega s} \cos^i s \sin^j s \, ds \\
& + c^-_{\ell,ijk_1\dots k_m 1_\omega} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s \sin^j s \, ds \Bigg),
\end{aligned}$$

for $1 \le \ell \le m$ and $m+1 \le \omega \le d$.

From (21) we get

$$\begin{aligned}
\gamma_\omega(\nu) = & \frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{P=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \Bigg(c^+_{\omega,ijk_1\dots k_m 0} \int_0^\phi e^{-s} \cos^i s \, \sin^j s \, ds \\
& + c^-_{\omega,ijk_1\dots k_m 0} e^{-2\pi} \int_\phi^{2\pi} e^{-s} \cos^i s \, \sin^j s \, ds \Bigg),
\end{aligned}$$

for $m+1 \le \omega \le d$.

From the above equalities and (22) we obtain for $1 \le \ell \le m$ that

$$\begin{aligned}
\widetilde{G}_{10}(\nu) = & \sum_{\omega=m+1}^{d} \Bigg[\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \Bigg(a^+_{ijk_1\dots k_m 1_\omega} \int_0^\phi e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds \\
& + b^+_{ijk_1\dots k_m 1_\omega} \int_0^\phi e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds + a^-_{ijk_1\dots k_m 1_\omega} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds \\
& + b^-_{ijk_1\dots k_m 1_\omega} \int_\phi^{2\pi} e^{\mu_\omega s} \cos^i s \, \sin^{j+1} s \, ds \Bigg) \Bigg] \Bigg[\frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \\
& \Bigg(c^+_{\omega,ijL_1\dots L_m 0} \int_0^\phi e^{-s} \cos^i s \, \sin^j s \, ds + c^-_{\omega,ijL_1\dots L_m 0} e^{-2\pi} \int_\phi^{2\pi} e^{-s} \cos^i s \, \sin^j s \, ds \Bigg) \Bigg],
\end{aligned}$$

$$
\begin{aligned}
\widetilde{G}_{1\ell}(\nu) = & \sum_{\omega=m+1}^{d} \Bigg[\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \Bigg(c^{+}_{\ell,ijk_1\dots k_m 1_\omega} \int_0^{\phi} e^{\mu_\omega s} \cos^i s \sin^j s \, ds \\
& + c^{-}_{\ell,ijk_1\dots k_m 1_\omega} \int_{\phi}^{2\pi} e^{\mu_\omega s} \cos^i s \sin^j s \, ds \Bigg) \Bigg] \Bigg[\frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{L_1} \dots z_m^{L_m} \\
& \Bigg(c^{+}_{\omega,ijL_1\dots L_m 0} \int_0^{\phi} e^{-s} \cos^i s \, \sin^j s \, ds + \, c^{-}_{\omega,ijL_1\dots L_m 0} \, e^{-2\pi} \int_{\phi}^{2\pi} e^{-s} \cos^i s \, \sin^j s \, ds \Bigg) \Bigg].
\end{aligned}
$$

Now from (17) we compute

$$
\begin{aligned}
\frac{\partial F^{\pm}_{10}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu)) = & \frac{1}{r} \sum_{P=0}^{n} (i+j) \, r^{i+j} z_1^{k_1} \dots z_m^{k_m} \\
& \Big(a^{\pm}_{ijk_1\dots k_m 0} \cos^{i+1} s \sin^j s + b^{\pm}_{ijk_1\dots k_m 0} \cos^i s \sin^{j+1} s \Big), \\
\frac{\partial F^{\pm}_{10}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu)) = & \sum_{P=0}^{n} k_\rho \, r^{i+j} z_1^{k_1} \dots z_\rho^{k_\rho - 1} \dots z_m^{k_m} \\
& \Big(a^{\pm}_{ijk_1\dots k_m 0} \cos^{i+1} s \sin^j s + b^{\pm}_{ijk_1\dots k_m 0} \cos^i s \sin^{j+1} s \Big), \\
\frac{\partial F^{\pm}_{10}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu)) = & \sum_{P=1}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \\
& \Big(a^{\pm}_{ijk_1\dots k_m 1_\omega} \cos^{i+1} s \sin^j s + b^{\pm}_{ijk_1\dots k_m 1_\omega} \cos^i s \sin^{j+1} s \Big),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F^{\pm}_{1\ell}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu)) = & \frac{1}{r} \sum_{P=0}^{n} (i+j) \, r^{i+j} z_1^{k_1} \dots z_m^{k_m} \, c^{\pm}_{\ell,ijk_1\dots k_m 0} \cos^i s \sin^j s, \\
\frac{\partial F^{\pm}_{1\ell}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu)) = & \sum_{P=0}^{n} k_\rho \, r^{i+j} z_1^{k_1} \dots z_p^{k_\rho - 1} \dots z_m^{k_m} \, c^{\pm}_{\ell,ijk_1\dots k_m 0} \cos^i s \sin^j s, \\
\frac{\partial F^{\pm}_{1\ell}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu)) = & \sum_{P=1}^{n} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \, c^{\pm}_{\ell,ijk_1\dots k_m 1_\omega} \cos^i s \sin^j s,
\end{aligned}
$$

for $1 \le \ell \le m$, $1 \le \rho \le m$ and $m+1 \le \omega \le d$.

Note that when $m = d$ we do not consider the functions $\dfrac{\partial F^{\pm}_{10}}{\partial z_\omega}$ and $\dfrac{\partial F^{\pm}_{1\ell}}{\partial z_\omega}$.

Now from (14) and (18) we get

$$
\begin{aligned}
y_{10}^{\pm}(s,\mathbf{z}_\nu) &= \sum_{P=0}^{n} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\left(a^{\pm}_{ijk_1\dots k_m0} I_{(i+1,j,s)} + b^{\pm}_{ijk_1\dots k_m0} I_{(i,j+1,s)}\right),\\
y_{1\rho}^{\pm}(\theta,\mathbf{z}_\nu) &= \sum_{P=0}^{n} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\, c^{\pm}_{\rho,ijk_1\dots k_m0} I_{(i,j,s)},\\
y_{1\omega}^{\pm}(\theta,\mathbf{z}_\nu) &= \sum_{P=0}^{n} r^{i+j} z_1^{k_1}\dots z_m^{k_m}\, c^{\pm}_{\omega,ijk_1\dots k_m0} \int_0^s e^{\mu_\omega(s-\tau)}\cos^i\tau\,\sin^j s\,d\tau,
\end{aligned}
$$

for $1\le\rho\le m$ and $m+1\le\omega\le d$. Therefore

$$
\int \frac{\partial F_{10}^{\pm}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu))y_{10}^{\pm}(s,\mathbf{z}_\nu)\,ds = \frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n}(i+j)\,r^{i+j+p+q}\,z_1^{k_1+L1}\dots z_m^{k_m+L_m}
$$

$$
\Bigg(a^{\pm}_{ijk_1\dots k_m0}a^{\pm}_{pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\,I_{(p+1,q,s)}ds
$$

$$
+\,b^{\pm}_{ijk_1\dots k_m0}a^{\pm}_{pqL_1\dots L_m0}\int\cos^{i}s\sin^{j+1} s\,I_{(p+1,q,s)}ds
$$

$$
+\,a^{\pm}_{ijk_1\dots k_m0}b^{\pm}_{pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\,I_{(p,q,s)}ds
$$

$$
+\,b^{\pm}_{ijk_1\dots k_m0}b^{\pm}_{pqL_1\dots L_m0}\int\cos^{i}s\;\sin^{j+1} s\,I_{(p,q+1,s)}ds\Bigg),
$$

$$
\int \frac{\partial F_{10}^{\pm}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu))y_{1\rho}^{\pm}(s,\mathbf{z}_\nu)ds = \sum_{P=0}^{n}\sum_{Q=0}^{n}k_\rho r^{i+j+p+q}z_1^{k_1+L_1}\dots z_\rho^{k_\rho+L_\rho-1}\dots z_m^{k_m+L_m}
$$

$$
\Bigg(a^{\pm}_{ijk_1\dots k_m0}\,c^{\pm}_{\rho,pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\,I_{(p,q,s)}\,ds
$$

$$
+\,b^{\pm}_{ijk_1\dots k_m0}\,c^{\pm}_{\rho,pqL_1\dots L_m0}\textstyle\int\cos^{i}s\sin^{j+1} s\,I_{(p,q,s)}\,ds\Bigg),
$$

$$
\int \frac{\partial F_{10}^{\pm}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu))y_{1\omega}^{\pm}(s,\mathbf{z}_\nu)ds = \sum_{P=0}^{n-1}\sum_{Q=0}^{n}r^{i+j+p+q}z_1^{k_1+L_1}\,z_2^{k_2+L_2}z_3^{k_3+L_3}\dots z_m^{k_m+L_m}
$$

$$
\Bigg(a^{\pm}_{ijk_1\dots k_m1_\omega}c^{\pm}_{\omega,pqL_1\dots L_m0}\int\cos^{i+1}s\sin^j s\int_0^s e^{\mu_\omega(s-\tau)}\cos^p\tau\sin^q\tau d\tau ds
$$

$$
+\,b^{\pm}_{ijk_1\dots k_m1_\omega}c^{\pm}_{\omega,pqL_1\dots L_m0}\int\cos^{i}s\sin^{j+1} s\int_0^s e^{\mu_\omega(s-\tau)}\cos^p\tau\sin^q\tau d\tau ds\Bigg),
$$

$$\int \frac{\partial F^{\pm}_{1\ell}}{\partial r}(s,\varphi(s,\mathbf{z}_\nu))y^{\pm}_{10}(s,\mathbf{z}_\nu)ds = \frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n}(i+j)\, r^{i+j+p+q} z_1^{k_1+L_1} z_2^{k_2+L_2} ... z_m^{k_m+L_m}$$

$$\left(c^{\pm}_{\ell,ijk_1...k_m0} a^{\pm}_{pqL_1...L_m0} \int \cos^i s \sin^j s \, I_{(p+1,q,s)}\, ds \right.$$

$$\left. + c^{\pm}_{\ell,ijk_1...k_m0} b^{\pm}_{pqL_1...L_m0} \int \cos^i s \sin^j s \, I_{(p,q+1,s)}\, ds \right),$$

$$\int \frac{\partial F^{\pm}_{1\ell}}{\partial z_\rho}(s,\varphi(s,\mathbf{z}_\nu))y^{\pm}_{1\rho}(s,\mathbf{z}_\nu)ds = \sum_{P=0}^{n}\sum_{Q=0}^{n} k_\rho\, r^{i+j+p+q} z_1^{k_1+L_1} ... z_\rho^{k_\rho+L_\rho-1} ... z_m^{k_m+L_m}$$

$$c^{\pm}_{\ell,ijk_1...k_m0}\, c^{\pm}_{\rho,pqL_1...L_m0} \int \cos^i s \sin^j s \, I_{(p,q,s)} ds,$$

$$\int \frac{\partial F^{\pm}_{1\ell}}{\partial z_\omega}(s,\varphi(s,\mathbf{z}_\nu))y^{\pm}_{1\omega}(s,\mathbf{z}_\nu)ds = \sum_{P=0}^{n-1}\sum_{Q=0}^{n} r^{i+j+p+q} z_1^{k_1+L_1}\, z_2^{k_2+L_2} z_3^{k_3+L_3} ... z_m^{k_m+L_m}$$

$$c^{\pm}_{\ell,ijk_1...k_m1_\omega} c^{\pm}_{\omega,pqL_1...L_m0} \int \cos^i s \sin^j s \int_0^s e^{\mu_\omega(s-\tau)} \cos^p \tau \sin^q \tau d\tau ds,$$

for $1 \le \ell \le m$, $1 \le \rho \le m$ and $m+1 \le \omega \le d$.

Moreover, from (17) we get

$$\int_0^\phi F^{+}_{20}(s,\mathbf{z}_\nu)\, ds = \sum_{P=0}^{n} \alpha^{+}_{ijk_1...k_m0}\, r^{i+j} z_1^{k_1} \ldots z_m^{k_m}\, I_{(i+1,j,\phi)}$$

$$+ \sum_{P=0}^{n} \beta^{+}_{ijk_1...k_m0}\, r^{i+j} z_1^{k_1} \ldots z_m^{k_m}\, I_{(i,j+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1...k_m0} b^{+}_{pqL_1...L_m0}\, r^{i+j+p+q}\, z_1^{k_1+L_1} \ldots z_m^{k_m+L_m}\, I_{(i+p+2,j+q,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1...k_m0} a^{+}_{pqL_1...L_m0}\, r^{i+j+p+q}\, z_1^{k_1+L_1} \ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q+1,\phi)}$$

$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b^{+}_{ijk_1...k_m0} b^{+}_{pqL_1...L_m0}\, r^{i+j+p+q}\, z_1^{k_1+L_1} \ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q+1,\phi)}$$

$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a^{+}_{ijk_1...k_m0} b^{+}_{pqL_1...L_m0}\, r^{i+j+p+q}\, z_1^{k_1+L_1} \ldots z_m^{k_m+L_m}\, I_{(i+p,j+q+2,\phi)},$$

$$\int_{\phi}^{2\pi} F_{20}^{-}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \alpha_{ijk_1\ldots k_m0}^{-}\, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i+1,j,\phi)}$$
$$+\sum_{P=0}^{n} \beta_{ijk_1\ldots k_m0}^{-}\, r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i,j+1,\phi)}$$
$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a_{ijk_1\ldots k_m0}^{-} b_{pqL_1\ldots L_m0}^{-}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+2,j+q,\phi)}$$
$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a_{ijk_1\ldots k_m0}^{-} a_{pqL_1\ldots L_m0}^{-}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q+1,\phi)}$$
$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b_{ijk_1\ldots k_m0}^{-} b_{pqL_1\ldots L_m0}^{-}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q+1,\phi)}$$
$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a_{ijk_1\ldots k_m0}^{-} b_{pqL_1\ldots L_m0}^{-}\, r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p,j+q+2,\phi)},$$

$$\int_{0}^{\phi} F_{2\ell}^{+}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \gamma_{\ell,ijk_1\ldots k_m}^{+} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, I_{(i,j,\phi)}$$
$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b_{ijk_1\ldots k_m0}^{+} c_{\ell,pqL_1\ldots L_m0}^{+} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p+1,j+q,\phi)}$$
$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} a_{ijk_1\ldots k_m0}^{+} c_{\ell,pqL_1\ldots L_m0}^{+} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, I_{(i+p,j+q+1,\phi)},$$

$$\int_{\phi}^{2\pi} F_{2\ell}^{-}(s,\mathbf{z}_\nu)\,ds = \sum_{P=0}^{n} \gamma_{\ell,ijk_1\ldots k_m0}^{-} r^{i+j} z_1^{k_1}\ldots z_m^{k_m}\, J_{(i,j,\phi)}$$
$$-\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0}^{n} b_{ijk_1\ldots k_m0}^{-} c_{\ell,pqL_1\ldots L_m0}^{-} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p+1,j+q,\phi)}$$
$$+\frac{1}{r}\sum_{P=0}^{n}\sum_{Q=0} a_{ijk_1\ldots k_m0}^{-} c_{\ell,pqL_1\ldots L_m0}^{-} r^{i+j+p+q}\, z_1^{k_1+L_1}\ldots z_m^{k_m+L_m}\, J_{(i+p,j+q+1,\phi)},$$

for $1 \le \ell \le m$.

On the other hand when the perturbation is continuous that is, $\phi = 2\pi$, we have

$$\widetilde{G}_{10}(\nu) = \sum_{\omega=m+1}^{d} \Big[\sum_{P=0}^{n-1} r^{i+j} z_1^{k_1} \dots z_m^{k_m}$$
$$\Big(a^+_{ijk_1\dots k_m 1\omega} \int_0^{2\pi} e^{\mu_\omega s} \cos^{i+1} s \, \sin^j s \, ds + \, b^+_{ijk_1\dots k_m 1\omega} \int_0^{2\pi} e^{\mu_\omega s} \cos^i s \sin^{j+1} s \, ds \Big) \Big]$$
$$\Big[\frac{-1}{1-e^{-\mu_\omega 2\pi}} \sum_{Q=0}^{n} r^{i+j} z_1^{L_1} \dots z_m^{L_m} \; c^+_{\omega,ijL_1\dots L_m 0} \int_0^{2\pi} e^{-s} \cos^i s \, \sin^j s \, ds \Big],$$

and

$$\int_0^{2\pi} F^+_{20}(s, \mathbf{z}_\nu)\, ds = \sum_{P=0,\, i\, odd,\, j\, even}^{n} \alpha^+_{ijk_1\dots k_m 0} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \; I_{(i+1,j,2\pi)}$$
$$+ \sum_{P=0,\, i\, even,\, j\, odd}^{n} \beta^+_{ijk_1\dots k_m 0} r^{i+j} z_1^{k_1} \dots z_m^{k_m} \; I_{(i,j+1,2\pi)}$$
$$- \frac{1}{r} \sum_{i,j\, even, P=0}^{n} \sum_{p,q\, even, Q=0}^{n} a^+_{ijk_1\dots k_m 0} b^+_{pqL_1\dots L_m 0} r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$
$$- \frac{1}{r} \sum_{i,j\, odd, P=0}^{n} \sum_{p,q\, odd, Q=0}^{n} a^+_{ijk_1\dots k_m 0} b^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p+2,j+q,2\pi)}$$
$$+ \frac{1}{r} \sum_{i,j\, odd, P=0}^{n} \sum_{p,q\, even, Q=0}^{n} a^+_{ijk_1\dots k_m 0} a^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p+1,j+q+1,2\pi)}$$
$$+ \frac{1}{r} \sum_{i,j\, even, P=0}^{n} \sum_{p,q\, odd, Q=0}^{n} a^+_{ijk_1\dots k_m 0} a^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p+1,j+q+1,2\pi)}$$
$$- \frac{1}{r} \sum_{i,j\, odd, P=0}^{n} \sum_{p,q\, even, Q=0}^{n} b^+_{ijk_1\dots k_m 0} b^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p+1,j+q+1,2\pi)}$$
$$- \frac{1}{r} \sum_{i,j\, even, P=0}^{n} \sum_{p,q\, odd, Q=0}^{n} b^+_{ijk_1\dots k_m 0} b^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p+1,j+q+1,2\pi)}$$
$$+ \frac{1}{r} \sum_{i,j\, even, P=0}^{n} \sum_{p,q\, even, Q=0}^{n} a^+_{ijk_1\dots k_m 0} b^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p,j+q+2,2\pi)}$$
$$+ \frac{1}{r} \sum_{i,j\, odd, P=0}^{n} \sum_{p,q\, odd, Q=0}^{n} a^+_{ijk_1\dots k_m 0} b^+_{pqL_1\dots L_m 0} \, r^{i+j+p+q} z_1^{k_1+L_1} \dots z_m^{k_m+L_m} I_{(i+p,j+q+2,2\pi)}.$$

Acknowledgements

J. Llibre is partially supported by the MINECO/FEDER grants MTM2016-77278-P and MTM2013-40998-P, and an AGAUR grant 2009SGR-0410. D.D. Novaes is partially supported by a FAPESP grant 2016/11471-2. I.O. Zeli is partially supported by a FAPESP grant 2013/21078-8 and CAPES grant CSF/PVE-88881.030454/2013-01.

References

[1] A. Buică and J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. Math.* **128** (2004), 7–22.

[2] E.A. Barbashin, *Introduction to the Theory of Stability* (T. Lukes, Ed.), Noordhoff, Groningen, 1970.

[3] F. Bizzari, M. Storace and A.Colombo, Bifurcation analysis of an impact model for forest fire prediction, *Int. J. Bifurcation and Chaos* **18** (2008), 2275–2288.

[4] C. Chicone, Lyapunov–Schmidt reduction and Melnikov integrals for bifurcation of periodic solutions in coupled oscillators, *J. of Diff. Equations* **112** (1994), 407–447.

[5] A. Cima, J. Llibre and M.A. Teixeira, Limit cycles of some polynomial differential systems in dimension 2, 3 and 4, via averaging theory, *Appl. Anal.* **87** (2008), 149–764.

[6] S. Coombes, Neuronal networks with gap junctions: A study of piecewise linear planar neuron models, *SIAM Applied Mathematics* **7** (2008), 1101–1129.

[7] W. Fulton, *Algebraic curves*, Mathematics Lecture. Note Series, Benjamin, 1974.

[8] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Elsevier Academic Press, 2007.

[9] J. Harris and B. Ermentrout, Bifurcations in the Wilson-Cowan equations with nonsmooth firing rate, *SIAM J. Appl. Dyn. Syst.* **14**(1) (2015), 43–72.

[10] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Discrete and Continuous Dynamical Systems-A **32** (2012), 2147–2164.

[11] Yu. Ilyashenko, Centennial history of Hilbert's 16th problem, *Bull. (New Series) Amer. Math. Soc.* **39** (2002), 301–354.

[12] T. Ito, A Filippov solution of a system of differential equations with discontinuous right-hand sides, *Economic Letters* **4** (1979), 349–354.

[13] V. Krivan, On the Gause predator-prey model with a refuge: A fresh look at the history, *J. of Theoretical Biology* **274** (2011), 67–73.

[14] J. Llibre and A.C. Mereu *Limit cycles for discontinuous quadratic differential systems with two zones*, J. of Mathematical Analysis and Applications (Print), **413** (2013), 763–775.

[15] J. Llibre, M. Ordóñez and E. Ponce, *On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry*, Nonlinear Analysis Series B: Real World App. **14** (2013), 2002–2012.

[16] J. Llibre, A.C. Mereu and D.D. Novaes, Averaging theory for discontinuous piecewise differential systems, *J. Differential Equation* **258** (2015), 4007–4032.

[17] J. Llibre and D.D. Novaes, Improving the averaging theory for computing periodic solutions of the differential equations, *Zeitschrift für angewandte Mathematik und Physik* **66**(4) (2015), 1401–1412.

[18] J. Llibre, D.D. Novaes and M.A. Teixeira, Higher order averaging theorem for finding periodic solutions via Brouwer degree, *Nonlinearity* **27** (2014), 563–583.

[19] J. Llibre, M.A. Teixeira and I.O. Zeli, Birth of limit cycles for a class of continuous and discontinuous differential systems in $(d+2)$-dimension, *Dynamical Systems* **31**(3) (2016), 237–250.
[20] J. Llibre, E. Ponce and J. Ros, Algebraic determination of limit cycles in 3–dimensional piecewise linear differential systems, *Nonlinear Analysis* **74** (2011), 6712–6727.
[21] N. Minorski, *Nonlinear Oscillations*, Van Nostrand, New York, 1962.
[22] W. Nicola and S.A. Campbell, Nonsmooth bifurcations of mean field systems of two-dimensional integrate and fire neurons, *SIAM Journal on Applied Dynamical Systems* **15** (2016), 391–439.
[23] D.D. Novaes, On nonsmooth perturbations of nondegenerate planar centers, *Publicacions matemàtiques* (2014), 395–420.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona (Uab), 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Universidade Estadual de Campinas (Unicamp), C.P. 651, 13083–859 Campinas, São Paulo, Brazil

E-mail address: ddnovaes@ime.unicamp.br, irisfalkoliv@ime.unicamp.br