# Grüss-type inequality by means of a fractional integral

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**Abstract:** We use a fractional integral recently proposed to establish a generalization of Grüss-type integral inequalities. We prove two theorems about these inequalities and enunciate and prove other inequalities associated with this fractional operator.

**Keywords:** Fractional integral, Generalization inequalities of Grüss-type

## 1 Introduction

In 1935, G. Grüss proved the following integral inequality [1]:

$$\left|\frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx-\left(\frac{1}{b-a}\int_a^b f\left(x\right)dx\right)\left(\frac{1}{b-a}\int_a^b g\left(x\right)dx\right)\right| \leq \frac{\left(M-m\right)\left(P-p\right)}{4}, \tag{1}$$

where $f$ and $g$ are integrable functions defined on $[a,b]$ and which satisfy the conditions

$$m \leq f(x) \leq M, \quad p \leq g(x) \leq P; \quad m, M, p, P \in \mathbb{R}, \quad x \in [a,b]. \tag{2}$$

Grüss-type inequalities have some important applications, among which we mention difference equations, integral arithmetic mean and $h$-integral arithmetic mean [2, 3]. On the other hand, Grüss-type inequalities are also studied in spaces with inner product, giving rise to some applications for the Mellin transform of sequences and polynomials in Hilbert spaces [4].

There are other important inequalities using integer order integrals, among which we mention Hölder's inequality, Jensen's inequality, Minkowski's inequality and reverse Minkowski's inequality [5, 6, 7, 8, 9, 10, 11]. For such inequalities, as well as for the study of functions, integrals and norms, the space $L^p(a,b)$ of $p-$integrable functions has a particular importance. However, we use in this work the space of Lebesgue measurable functions which admits, as a particular case, $L^p(a,b)$.

The creation of fractional calculus gave rise to several results and important theories in mathematics, physics, engineering and other fields of science. As a result, it has been possible to define several different types of fractional integrals and derivatives such as the Riemann-Liouville, Katugampola, Hadamard, Erdélyi-Kober, Liouville and Weyl types. There are still other types of fractional integrals which can be found in [12]. On the other hand, even if we consider only integrals with a nonsingular kernel [13, 14], we can mention several recent papers with interesting applications: Yang [15] presents a class of fractional derivatives of constant and variable order associated with fractional order relaxation equations; Yang et al. [16] propose a new derivative to discuss the steady heat-conduction problem; Gao and Yang [17], using the Caputo-Fabrizio derivative, discuss the so-called fractional Maxwell fluid; and Yang and Tenreiro Machado [18], using an operator of variable order, propose a new formulation for Caputo's fractional derivative and apply it to the study of transport processes in complex media.

Over the years, some inequalities involving fractional integrals have been discovered, such as the reverse Minkowski, Hermite-Hadamard, Ostrowski and Fejér inequalities [8, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. We also mention that there are in the literature generalizations of Eq.(1) using, for example, fractional integrals of Riemann-Liouville, Hadamard and $q-$fractional types [29, 30, 31]. We remark that this latter type of fractional integral is a general one, because it unifies all six integrals mentioned above; moreover, it allows one to obtain a general Grüss-type inequality. This new, general Grüss-type inequality plays a role similar to that of the Grönwall inequality in the demonstration of the uniqueness of Cauchy's problem [32, 33].

In this paper, using a fractional integral recently introduced [34], we propose a new generalization of Eq.(1), i.e., new Grüss-type inequalities that generalize inequalities obtained by means of the Riemann-Liouville fractional integral [29].

The paper is organized as follows: In section 2 we present Katugampola's fractional integral, the space where we use it and the parameters necessary to recover the six fractional integrals mentioned above as particular cases. In section 3 we present several Grüss-type inequalities, our main result. In section 4 we discuss other inequalities involving Katugampola's fractional integral. Concluding remarks close the paper.

## 2 Preliminaries

In this section, we define the space $X_c^p(a,b)$, where the Katugampola's fractional integrals are defined. With a convenient choice of parameters, we recover six well-known fractional integrals, previously mentioned [12, 34].

**Definition 1.** *The space $X_c^p(a,b)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$ consists of those complex-valued Lebesgue measurable functions $\varphi$ on $(a,b)$ for which $\|\varphi\|_{X_c^p} < \infty$, with*

$$\|\varphi\|_{X_c^p} = \left( \int_a^b |x^c \varphi(x)|^p \frac{dx}{x} \right)^{1/p} \quad (1 \leq p < \infty)$$

*and*

$$\|\varphi\|_{X_c^\infty} = \sup \mathrm{ess}_{x \in (a,b)} \, [x^c |\varphi(x)|].$$

In particular, when $c = 1/p$, the space $X_c^p(a,b)$ coincides with the space $L^p(a,b)$.

**Definition 2.** *Let $\varphi \in X_c^p(a,b)$, $\alpha > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then, the fractional integrals of a function $\varphi$, left- and right-sided, are respectively defined by*

$$^{\rho}\mathcal{I}_{a+,\eta,\kappa}^{\alpha,\beta} \varphi(x) := \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau)\, d\tau, \;\; 0 \leq a < x < b \leq \infty \tag{3}$$

*and*

$$^{\rho}\mathcal{I}_{b-,\eta,\kappa}^{\alpha,\beta} \varphi(x) := \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \varphi(\tau)\, d\tau, \quad 0 \leq a < x < b \leq \infty, \tag{4}$$

*if the integrals exist.*

As previously mentioned, with an adequate choice of parameters, the fractional integral given by Eq.(3) admits, as particular cases, six well-known fractional integrals, namely:

1. If $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 1$ in Eq.(3), we obtain the Riemann-Liouville fractional integral [12, p. 69].

2. If $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, taking the limit $\rho \to 0^+$ and using l'Hospital's rule, in Eq.(3), we obtain the Hadamard fractional integral [12, p. 110].

3. When $\beta = 0$ and $\kappa = -\rho(\alpha + \eta)$, in Eq.(3), we obtain the Erdélyi-Kober fractional integral [12, p. 105].

4. Also, for $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in Eq.(3), we recover the Katugampola fractional integral [35].

5. With the choice $\kappa = 0$, $\eta = 0$, $a = -\infty$ and taking the limit $\rho \to 1$ in Eq.(3), we have the Weyl fractional integral [36, p. 50].

6. If $\kappa = 0$, $\eta = 0$, $a = 0$ and taking the limit $\rho \to 1$ in Eq.(3), we obtain the Liouville fractional integrals [12, p. 79].

Notice that we present and discuss our new results associated with the fractional integral using the left-sided operator only. Moreover, we admit $a = 0$ in Eq.(3), in order to have

$$ {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}\varphi\left(x\right) = \frac{\rho^{1-\beta}x^{\kappa}}{\Gamma\left(\alpha\right)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{\left(x^{\rho}-\tau^{\rho}\right)^{1-\alpha}}\varphi\left(\tau\right)d\tau. \tag{5} $$

# 3 Main results

We begin enunciating and proving Lemma 1 in order to use it in the first theorem that generalizes the inequalities of Grüss type, Eq.(1).

First, let $\alpha > 0$, $x > 0$ and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. We define the function

$$ \Lambda_{x,\kappa}^{\rho,\beta}\left(\alpha,\eta\right) = \frac{\Gamma\left(\eta+1\right)}{\Gamma\left(\eta+\alpha+1\right)}\rho^{-\beta}x^{\kappa+\rho(\eta+\alpha)}, \tag{6} $$

in order to simplify the development and notation.

**Lemma 1.** *Let $m, M, \beta, \kappa \in \mathbb{R}$ and let $u$ be an integrable function on $[0,\infty)$. Then, for all $x > 0$, $\alpha > 0$, $\rho > 0$ and $\eta \geq 0$ we have*

$$ \begin{aligned} &\Lambda_{x,\kappa}^{\rho,\beta}\left(\alpha,\eta\right)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u^{2}(x) - \left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)\right)^{2} \\ &= \left(M\Lambda_{x,\kappa}^{\rho,\beta}\left(\alpha,\eta\right) - {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)\right)\times\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x) - m\Lambda_{x,\kappa}^{\rho,\beta}\left(\alpha,\eta\right)\right) \\ &-\Lambda_{x,\kappa}^{\rho,\beta}\left(\alpha,\eta\right)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}(M-u(x))(u(x)-m), \end{aligned} \tag{7} $$

*with $\Lambda_{x,\kappa}^{\rho,\beta}\left(\alpha,\eta\right)$ given by* Eq.(6).

***Proof.*** *Let $m, M \in \mathbb{R}$ and let $u$ be an integrable function on $[0,\infty)$. For all $\tau, \xi \in [0,\infty)$ we have*

$$ \begin{aligned} &(M-u(\xi))(u(\tau)-m) + (M-u(\tau))(u(\xi)-m) - (M-u(\tau))(u(\tau)-m) \\ &-(M-u(\xi))(u(\xi)-m) = u^{2}(\tau) + u^{2}(\xi) - 2u(\tau)u(\xi). \end{aligned} \tag{8} $$

*Multiplying both sides of* Eq.(8) *by* $\dfrac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\alpha)}\dfrac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}$*, where* $\tau\in(0,x)$*,* $x>0$*, and integrating with respect to variable* $\tau$ *from* 0 *to* $x$*, we obtain*

$$\begin{aligned}&(M-u(\xi))\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)-m\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right)+(u(\xi)-m)\\&\times\left(M\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)\right)-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}(M-u(x))(u(x)-m)\right)\\&-(M-u(\xi))(u(\xi)-m)\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)={}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u^2(x)\\&+u^2(\xi)\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-2u(\xi){}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x).\end{aligned}\tag{9}$$

*Also, multiplying* Eq.(9) *by* $\dfrac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\alpha)}\dfrac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\alpha}}$*, where* $\xi\in(0,x)$ *and* $x>0$*, and integrating with respect to variable* $\xi$ *from* 0 *to* $x$*, we get*

$$\begin{aligned}&\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)-m\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right)\left(M\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)\right)\\&+\left(M\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)\right)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)-m\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right)\\&-\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\;{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}(M-u(x))(u(x)-m)\\&-\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\;{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}(M-u(x))(u(x)-m)\\&=\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\;{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u^2(x)+\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u^2(x)\\&-2{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}u(x),\end{aligned}\tag{10}$$

*where we have introduced* $\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)$ *in* Eq.(6).

*Rearranging* Eq.(10)*, we immediately get* Eq.(7). □

Note that when $\eta=0$, $\kappa=0$ and $\rho\to1$ in Eq.(7), we have

$$\begin{aligned}&\left(M\frac{x^{\alpha}}{\Gamma(\alpha+1)}-I^{\alpha}u(x)\right)\left(I^{\alpha}u(x)-m\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)\\&-\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}-I^{\alpha}(M-u(x))(u(x)-m)\right)\\=\;&\frac{x^{\alpha}}{\Gamma(\alpha+1)}-I^{\alpha}u^2(x)-(I^{\alpha}u(x))^2,\end{aligned}$$

where

$$\lim_{\rho\to1}\Lambda^{\rho,\beta}_{x,0}(\alpha,0)=\Lambda_x(\alpha,0)=\frac{x^{\alpha}}{\Gamma(\alpha+1)}.$$

This result was obtained in [29] using the Riemann-Liouville fractional integral.

**Theorem 1.** *Let* $f$ *and* $g$ *be two integrable functions on* $[0,\infty)$ *and such that*

$$m\le f(x)\le M\quad\text{and}\quad p\le g(x)\le P\quad\text{for}\quad m,M,p,P\in\mathbb{R}\quad\text{and}\quad x\in[0,\infty).\tag{11}$$

*Then, for all* $\beta,\kappa\in\mathbb{R}$*,* $x>0$*,* $\alpha>0$*,* $\rho>0$ *and* $\eta\ge0$*, we have*

$$\left|\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\;{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg(x)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right|\le\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right)^2(M-m)(P-p).\tag{12}$$

***Proof.*** *Let* $f$ *and* $g$ *be two integrable functions satisfying* Eq.(11)*. Consider*

$$H(\tau,\xi)=(f(\tau)-f(\xi))(g(\tau)-g(\xi));\quad\tau,\xi\in(0,x)\quad\text{and}\quad x>0,$$

*which can be written as*

$$H(\tau,\xi)=f(\tau)g(\tau)-f(\tau)g(\xi)-f(\xi)g(\tau)+f(\xi)g(\xi). \tag{13}$$

*Multiplying both sides of* Eq.(13) *by* $\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}$*, where* $\tau\in(0,x)$*,* $x>0$*, and integrating with respect to variable* $\tau$ *from* $0$ *to* $x$*, we obtain*

$$\begin{aligned}&\frac{\rho^{\beta-1}x^{\kappa}}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}H(\tau,\xi)d\tau={}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg(x)-g(\xi)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\\&-f(\xi)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)+\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,f(\xi)g(\xi).\end{aligned} \tag{14}$$

*Multiplying* Eq.(14) *by* $\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\alpha)}\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\alpha}}$*, with* $\xi\in(0,x)$*,* $x>0$*, and integrating with respect to variable* $\xi$ *from* $0$ *to* $x$*, we have*

$$\begin{aligned}&\frac{\rho^{2(1-\beta)}x^{2\kappa}}{\Gamma^2(\alpha)}\int_0^x\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\alpha}}H(\tau,\xi)d\tau d\xi\\=&\;2\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg(x)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right).\end{aligned}$$

*Applying the Cauchy-Schwarz inequality* [37]*, we can write*

$$\begin{aligned}&\left|\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg(x)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right|^2\\&\leq\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f^2(x)-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\right)^2\right)\\&\times\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g^2(x)-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right)^2\right).\end{aligned} \tag{15}$$

*Since* $(M-f(x))(f(x)-m)\geq0$ *and* $(P-g(x))(g(x)-P)\geq0$*, we have*

$$\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}(M-f(x))(f(x)-m)\geq0 \tag{16}$$

*and*

$$\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}(P-g(x))(g(x)-P)\geq0. \tag{17}$$

*Thus,*

$$\begin{aligned}&\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f^2(x)-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\right)^2\right)\leq\left(M\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\right)\\&\times\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)-m\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right)\end{aligned} \tag{18}$$

*and*

$$\begin{aligned}&\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g^2(x)-\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right)^2\right)\leq\left(P\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right)\\&\times\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)-p\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right).\end{aligned} \tag{19}$$

*Combining* Eq.(15)*,* Eq.(18) *and* Eq.(19) *and using* Lemma 1*, we conclude that*

$$\begin{aligned}&\left(\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg(x)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\,{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right)^2\\&\leq\left(M\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)\right)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f(x)-m\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right)\\&\times\left(P\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)-{}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)\right)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g(x)-p\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right).\end{aligned} \tag{20}$$

*Further, using the inequality* $4ab \leq (a+b)^2$, $a, b \in \mathbb{R}$, *we have*

$$4\left(M\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta) - {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\right)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x) - m\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\right)$$
$$\leq \left(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)(M-m)^2\right), \tag{21}$$

*and*

$$4\left(P\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta) - {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\right)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x) - p\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\right)$$
$$\leq \left(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)(P-p)^2\right). \tag{22}$$

*Finally, from* Eq.(20), Eq.(21) *and* Eq.(22), *we obtain* Eq.(12). □

Applying Theorem 1 for $\alpha = 1$, $\rho \to 1$, $\eta = 0$ and $\kappa = 0$, we obtain the classical inequality of Grüss type, Eq.(1). On the other hand, for $\rho \to 1$, $\eta = 0$ and $\kappa = 0$, we recover the Theorem 3.1 in [29].

In order to prove the next theorem, which generalizes the inequalities of Grüss type, we need the following lemmas, in which we have used the notation of Eq.(6) with $\alpha = \gamma$, i.e., $\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)$.

**Lemma 2.** *Let* $f$ *and* $g$ *be two integrable funtions on* $[0,\infty)$. *Then, for all* $\beta, \kappa \in \mathbb{R}$, $x > 0$, $\alpha > 0$, $\rho > 0$, $\eta \geq 0$ *and* $\gamma > 0$ *we have*

$$\Big(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}fg(x) + \Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg(x)$$
$${}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x) - {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\Big)^2$$
$$\leq \Big(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f^2(x) + \Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f^2(x)$$
$$-2\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)\Big)\Big(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g^2(x)$$
$$+\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g^2(x) - 2\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)\Big). \tag{23}$$

***Proof.*** *Multiplying both sides of* Eq.(14) *by* $\dfrac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\dfrac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}$, *where* $\xi \in (0,x)$, $x > 0$, *and integrating with respect to variable* $\xi$ *from* $0$ *to* $x$, *we obtain*

$$\begin{aligned}
&\frac{\rho^{2(1-\beta)}x^{2\kappa}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^x\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}H(\tau,\xi)\,d\tau\, d\xi \\
=\ &\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg(x) + \Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}fg(x) \\
&-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x) - {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x).
\end{aligned}$$

*Using the Cauchy-Schwarz inequality, we have*

$$\begin{aligned}
&\Big|\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg(x) + \Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}fg(x) \\
&- {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x) - {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)\Big|^2 \\
\leq\ &\Big(\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f^2(x) + \Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f^2(x) \\
&-2\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)\Big)\Big(\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g^2(x) \\
&\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g^2(x) - 2\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)\Big),
\end{aligned}$$

*which is* Eq.(23). □

**Lemma 3.** *Let $u$ be an integrable function on $[0,\infty)$ satisfying* Eq.(11) *on $[0,\infty)$. Then, for all $\beta,\kappa\in\mathbb{R}$, $x>0$, $\alpha>0$, $\rho>0$, $\eta\geq 0$ and $\gamma>0$ we have*

$$\begin{aligned}
&\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}u^2(x)+\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u^2(x)\\
&-2\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}u(x)=\left(M\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)\right)\\
&\times\left({}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}u(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\right)+\left(M\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)-{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}u(x)\right)\\
&\times\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\right)-\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}(M-u(x))(u(x)-m)\\
&-\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}(M-u(x))(u(x)-m).
\end{aligned}\tag{24}$$

***Proof.*** *Multiplying both sides of* Eq.(9) *by* $\dfrac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\dfrac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}$, *where $\xi\in(0,x)$, $x>0$, and integrating with respect to variable $\xi$ from 0 to $x$, we obtain*

$$\begin{aligned}
&\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\right)\left(\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}(M-u(\xi))d\xi\right)\\
&+\left(M\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)\right)\left(\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}(u(\xi)-m)d\xi\right)\\
&-\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}(M-u(x))(u(x)-m)\,\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}d\xi\right)\\
&-\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ \frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}(M-u(x))(u(x)-m)d\xi\\
=\ &{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u^2(x)\left(\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}d\xi\right)\\
&+\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\left(\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}u^2(\xi)d\xi\right)\\
&-2\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}u(x)\left(\frac{\rho^{1-\beta}\,x^{\kappa}}{\Gamma(\gamma)}\int_0^x\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\gamma}}u(\xi)d\xi\right).
\end{aligned}$$

*From this last expression,* Eq.(24) *follows immediately.* □

Considering $\eta=0$, $\kappa=0$ and $\rho\to 1$ in Lemma 2 and Lemma 3, we obtain the results of Lemma 3.4 and Lemma 3.5 in [29].

We use Lemma 2 and Lemma 3 to prove the next theorem.

**Theorem 2.** *Let $f$ and $g$ be two integrable functions on $[0,\infty)$ satisfying the conditions in* Eq.(11) *on $[0,\infty)$. Then, for all $\beta,\kappa\in\mathbb{R}$, $x>0$, $\alpha>0$, $\gamma>0$ and $\eta\geq 0$ we have*

$$\begin{aligned}
&\Big(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}fg(x)+\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\ {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg(x)\\
&-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)-{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\Big)^2\\
\leq\ &\Big[\left(M\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\right)\left({}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\right)\\
&+\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\right)\left(M\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)-{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)\right)\Big]\\
&\times\Big[\left(P\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\right)\left({}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)-p\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\right)\\
&+\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)-p\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\right)\left(P\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)-{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)\right)\Big].
\end{aligned}\tag{25}$$

***Proof.*** *Since $(M-f(x))(f(x)-m)\geq 0$, $(P-g(x))(g(x)-p)\geq 0$, $x>0$ and $\rho>0$, we can write*

$$\begin{aligned}&-\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}(M-f(x))(f(x)-m)\\&-\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}(M-f(x))(f(x)-m)\leq 0\end{aligned} \tag{26}$$

*and*

$$\begin{aligned}&-\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}(P-g(x))(g(x)-p)\\&-\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}(P-g(x))(g(x)-p)\leq 0.\end{aligned} \tag{27}$$

*Applying* Lemma 3 *for $f$ and $g$ and using* Eq.(26) *and* Eq.(27)*, we obtain*

$$\begin{aligned}&\big(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f^2(x)+\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f^2(x)\\&-2\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\big)\leq\big(M\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)\big)\\&\times\big({}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\big)+\big({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f(x)-m\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\big)\\&\big(M\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)-{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}f(x)\big)\end{aligned} \tag{28}$$

*and*

$$\begin{aligned}&\big(\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g^2(x)+\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g^2(x)\\&-2\,{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}\,{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\big)\leq\big(P\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)-{}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)\big)\\&\times\big({}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)-p\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)\big)+\big({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g(x)-p\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\big)\\&\big(P\Lambda_{x,\kappa}^{\rho,\beta}(\gamma,\eta)-{}^{\rho}I_{\eta,\kappa}^{\gamma,\beta}g(x)\big)\end{aligned} \tag{29}$$

*Taking the product of* Eq.(28) *by* Eq.(29) *and using* Lemma 2*,* Eq.(25) *follows immediately.* □

Taking $\alpha=\gamma$ in Theorem 2, we obtain Theorem 1. On the other hand, considering $\eta=0$, $\kappa=0$ and $\rho\to 1$ in Theorem 2, we obtain Theorem 3.3 of [29]. Considering Theorem 2 with $\alpha=\gamma=1$, $\eta=0$, $\kappa=0$ and $\rho\to 1$, we recover Eq.(1).

We mention that the results obtained in Lemmas 1, 2, 3 and Theorems 1 and 2 can be proved considering the integral in Eq.(5), from $a=1$ to $x$, in order to obtain, as particular cases, the generalized inequalities of Grüss type discussed in [30], with the Hadamard fractional integral.

# 4 Other fractional integral inequalities

In this section, we present some integral inequalities involving Katugampola's fractional operator. The results obtained were adapted from the paper by Chinchane & Pachpatte [30], in which the authors present a brief approach to Grüss-type inequalities, using the Hadamard fractional integral.

Before we begin, it should be emphasized that, although the results presented here have been adapted for the left-sided Katugampola operator, Eq.(5), it is possible to obtain similar results for Hadamard's operator by means of an adequate choice of the operator's parameters. Indeed, the results presented can be reproduced for Hadamard's operator by putting $a=1$ in Eq.(5).

We now present some theorems involving fractional integrals inequalities.

**Theorem 3.** *Let $\alpha > 0$, $\beta, \rho, \eta, \kappa \in \mathbb{R}$; let $f, g \in X_c^p(0,x)$ be two positive functions defined on $[0,\infty)$, $x > 0$ and $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, the following inequalities hold:*

1. $\dfrac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^p(x)}{p} + \dfrac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^q(x)}{q} \geq \dfrac{\Gamma(\eta+\alpha+1)\rho^{\beta}}{\Gamma(\eta+1)\, x^{\rho(\eta+\alpha)+\kappa}} \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g(x)\right).$

2. $\dfrac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^p(x) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^p(x)}{p} + \dfrac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^q(x) {}^{\rho}I_{\eta,\kappa}^{\alpha,\beta} g^q(x)}{q} \geq \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right)^2.$

3. $\dfrac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^p(x) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^q(x)}{p} + \dfrac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^q(x) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^p(x)}{q} \geq \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} (fg)^{p-1}(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} (fg)^{q-1}(x)\right).$

4. ${}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^p(x) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^q(x) \geq \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^{p-1}(x)\, g^{q-1}(x)\right).$

**Proof.** 1. Considering the Young inequality [38],

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \ \forall a, b \geq 0, \ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{30}$$

and putting $a = f(t)$ and $b = f(s)$, $s > 0$, in the inequality of Eq.(30), we have

$$\frac{f^p(t)}{p} + \frac{g^q(s)}{q} \geq f(t)\, g(s), \ \forall f(t)\, g(s) \geq 0. \tag{31}$$

Multiplying both sides of Eq.(31) by $\dfrac{\rho^{1-\beta} x^{\kappa} t^{\rho(\eta+1)-1}}{\Gamma(\alpha)\,(x^{\rho} - t^{\rho})^{1-\alpha}}$ and integrating with respect to variable $t$ on $\in (0,x)$, $x > 0$, we have

$$\begin{aligned}
&\frac{\rho^{1-\beta} x^{\kappa}}{p\Gamma(\alpha)} \int_0^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} f^p(t)\, dt + \frac{g^q(s)\, \rho^{1-\beta} x^{\kappa}}{q\Gamma(\alpha)} \int_0^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} dt \\
\geq \ &\frac{g(s)\, \rho^{1-\beta} x^{\kappa}}{p\Gamma(\alpha)} \int_0^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} f(t)\, dt,
\end{aligned} \tag{32}$$

which can be rewritten as

$$\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^p(x)}{p} + \frac{g^q(s)\, \rho^{1-\beta} x^{\kappa}}{q\Gamma(\alpha)} \int_0^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} dt \geq g(s) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x). \tag{33}$$

Further, with the change of variable $u = \frac{t^{\rho}}{x^{\rho}}$ in the integral $\displaystyle\int_0^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} dt$, we obtain

$$\int_0^x \frac{t^{\rho(\eta+1)-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} dt = \int_0^1 \frac{u^{\eta} x^{\rho\eta}}{x^{\rho(1-\alpha)}(1-u)^{1-\alpha}} \frac{x^{\rho}}{\rho} du = \frac{x^{\rho(\eta+\alpha)}}{\rho} B(\eta+1, \alpha), \tag{34}$$

where $B(a,b)$ is the beta function. Using the identity $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ in Eq.(34) and replacing the result in Eq.(33), we have

$$\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^p(x)}{p} + \frac{g^q(s)\, x^{\rho(\eta+\alpha)+\kappa}}{q\rho^{\beta}} \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \geq g(s) {}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x). \tag{35}$$

Multiplying both sides of Eq.(35) by $\dfrac{\rho^{1-\beta}s^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-s^{\rho})^{1-\alpha}}$ and integrating with respect to the variable $s$ on $\in(0,x)$, $x>0$, we have

$$\begin{aligned}
&\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}(x)}{p\Gamma(\alpha)}\rho^{1-\beta}x^{\kappa}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}}ds+\frac{\Gamma(\eta+1)\,x^{\rho(\eta+\alpha)+2\kappa}\rho^{1-\beta}}{\Gamma(\alpha)\Gamma(\eta+\alpha+1)\,q\rho^{\beta}}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}}g^{q}(s)\,ds\\
\geq\ &\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f(x)\right)\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}}g(s)\,ds,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}(x)\,\rho^{1-\beta}x^{\kappa}}{p\Gamma(\alpha)}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}ds+\frac{x^{\rho(\eta+\alpha)+\kappa}\Gamma(\eta+1)}{q\rho^{\beta}\Gamma(\eta+\alpha+1)}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}(x)\right)\\
\geq\ &\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f(x)^{\rho}\,\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g(x)\right). \qquad (36)
\end{aligned}$$

With the change of variable $u=\frac{s^{\rho}}{x^{\rho}}$ in integral $\displaystyle\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}ds$, we obtain

$$\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-t^{\rho})^{1-\alpha}}ds=\frac{x^{\rho(\eta+\alpha)}\Gamma(\eta+1)\Gamma(\alpha)}{\rho\Gamma(\eta+\alpha+1)}. \qquad (37)$$

Substituting Eq.(37) into Eq.(36), we conclude that

$$\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}(x)}{p}+\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{p}(x)}{q}\geq\frac{\Gamma(\eta+\alpha+1)\,\rho^{\beta}}{\Gamma(\eta+1)\,x^{\rho(\eta+\alpha)+\kappa}}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f(x)^{\rho}\,\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g(x)\right).$$

2. For the proof of item (2), we take $a=f(t)\,g(s)$ and $b=f(s)\,g(t)$, replace them in Young's inequality and follow the same steps used in the proof of item (1).

3. In order to prove item (3) it is enough to take $a=\frac{f(t)}{g(t)}$ and $b=\frac{f(s)}{g(s)}$ in Young's inequality and proceed as in item (1).

4. Putting $a=\frac{f(s)}{f(t)}$ and $b=\frac{g(s)}{g(t)}$, $f(t),g(t)\neq 0$, in Eq.(30), we get

$$\frac{f^{p}(t)}{p}+\frac{g^{q}(s)}{q}\geq f(t)\,g(s)\,,\ \forall f(t)\,g(s)\geq 0. \qquad (38)$$

Multiplying both sides of Eq.(38) by $\dfrac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ and integrating with respect to variable $t$ on $(0,x)$, $x>0$, we have

$$f^{p}(s)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}(x)\right)+g^{q}(s)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}(x)\right)\geq f(s)\,g(s)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p-1}(x)\,g^{q-1}(x)\right). \qquad (39)$$

Multiplying both sides of Eq.(39) by $\dfrac{\rho^{1-\beta}x^{\kappa}s^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ and integrating with respect to variable $s$ on $(0,x)$, $x>0$, we get

$$\begin{aligned}
&\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}(x)\,\rho^{1-\beta}x^{\kappa}}{p\Gamma(\alpha)}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}}f^{p}(s)\,ds+\frac{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}(x)\,\rho^{1-\beta}x^{\kappa}}{q\Gamma(\alpha)}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}}g^{q}(s)\,ds\\
\geq\ &\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p-1}(x)\,g^{q-1}(x)\right)\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_{0}^{x}\frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}}f(s)\,g(s)\,ds. \qquad (40)
\end{aligned}$$

Using the identity $\frac{1}{p}+\frac{1}{q}=1$ in Eq.(40), we conclude that

$$^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}\left(x\right)\geq\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f\left(x\right)g\left(x\right)\right)\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p-1}\left(x\right)g^{q-1}\left(x\right)\right),$$

which is item (4). □

**Theorem 4.** *Let $\alpha>0$, $\beta,\rho,\eta,\kappa\in\mathbb{R}$ and let $f,g\in X_{c}^{p}(0,x)$ be two positive functions on $[0,\infty)$, $x>0$; let $p,q>1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. Then, the following inequalities hold:*

1. $\dfrac{^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}\left(x\right)}{p}+\dfrac{^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}\left(x\right)}{q}\geq\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f\left(x\right)g\left(x\right)\right)\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{\frac{2}{q}}\left(x\right)g^{\frac{2}{p}}\left(x\right)\right).$

2. $\dfrac{^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}\left(x\right)}{p}+\dfrac{^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{q}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}\left(x\right)}{q}\geq\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{\frac{2}{q}}\left(x\right)g^{\frac{2}{p}}\left(x\right)\right)\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p-1}\left(x\right)g^{q-1}\left(x\right)\right).$

3. $^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}\left(\dfrac{g^{p}\left(x\right)}{p}+\dfrac{g^{q}\left(x\right)}{q}\right)\geq\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{\frac{2}{p}}\left(x\right)g\left(x\right)\right)\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{\frac{2}{q}}\left(x\right)g\left(x\right)\right).$

**Proof.** 1. Substituting $a=f\left(t\right)g^{\frac{2}{p}}\left(s\right)$ and $b=f^{\frac{2}{q}}\left(s\right)g\left(t\right)$ into Eq.(30), we have

$$\frac{f^{p}\left(t\right)g^{2}\left(s\right)}{p}+\frac{g^{q}\left(s\right)f\left(s\right)}{q}\geq f\left(t\right)g\left(s\right)f^{\frac{2}{p}}\left(t\right)g^{\frac{2}{q}}\left(s\right). \tag{41}$$

Multiplying both sides of Eq.(41) by $\dfrac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}$ and integrating with respect to variable $t$ on $(0,x)$, $x>0$, we have

$$\frac{g^{2}\left(s\right)}{p}\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}\left(x\right)\right)+\frac{f^{2}\left(s\right)}{q}\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}\left(x\right)\right)\leq g^{\frac{2}{p}}\left(s\right)f^{\frac{2}{q}}\left(s\right)\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f\left(x\right)g\left(x\right)\right). \tag{42}$$

Now, multiplying both sides of Eq.(42) by $\dfrac{\rho^{1-\beta}x^{\kappa}s^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-s^{\rho}\right)^{1-\alpha}}$ and integrating with respect to variable $s$ on $(0,x)$, $x>0$, we conclude that

$$\frac{^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}\left(x\right)}{p}+\frac{^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}\left(x\right)^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}\left(x\right)}{q}\geq\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f\left(x\right)g\left(x\right)\right)\left(^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{\frac{2}{p}}\left(x\right)f^{\frac{2}{q}}\left(x\right)\right).$$

2. Taking $a=\frac{f^{\frac{2}{p}}(t)}{f(s)}$ and $b=\frac{g^{\frac{2}{q}}(t)}{g(s)}$, with $f\left(s\right),g\left(s\right)\neq0$, and substituting into Eq.(30), we have

$$\frac{f^{2}\left(t\right)}{pf^{p}\left(s\right)}+\frac{g^{2}\left(t\right)}{pg^{p}\left(s\right)}\geq\frac{f^{\frac{2}{p}}\left(t\right)g^{\frac{2}{q}}\left(t\right)}{f\left(s\right)g\left(s\right)},$$

which can be rewritten as

$$\frac{f^{2}\left(t\right)g^{q}\left(s\right)}{p}+\frac{g^{2}\left(t\right)f^{p}\left(s\right)}{q}\geq f^{p-1}\left(s\right)g^{q-1}\left(s\right)f^{\frac{2}{p}}\left(t\right)g^{\frac{2}{q}}\left(t\right). \tag{43}$$

Again multiplying by $\dfrac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma\left(\alpha\right)\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}$ both sides of Eq.(43), and integrating with respect to the variable $t$ on $(0,x)$, $x>0$, we have

$$\begin{aligned}&\frac{g^{q}\left(s\right)\rho^{1-\beta}x^{\kappa}}{p\Gamma\left(\alpha\right)}\int_{0}^{x}\frac{t^{\rho(\eta+1)-1}}{\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}f^{2}\left(t\right)dt+\frac{f^{p}\left(s\right)\rho^{1-\beta}x^{\kappa}}{q\Gamma\left(\alpha\right)}\int_{0}^{x}\frac{t^{\rho(\eta+1)-1}}{\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}g^{2}\left(t\right)dt\\ \geq\ &\frac{f^{p-1}\left(s\right)g^{q-1}\rho^{1-\beta}x^{\kappa}}{\Gamma\left(\alpha\right)}\int_{0}^{x}\frac{t^{\rho(\eta+1)-1}}{\left(x^{\rho}-t^{\rho}\right)^{1-\alpha}}f^{\frac{2}{p}}\left(t\right)g^{\frac{2}{q}}\left(t\right)dt,\end{aligned}$$

or in the following form:

$$\frac{g^{q}(s)}{p}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}(x)\right)+\frac{f^{p}(s)}{q}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}(x)\right)\geq f^{p-1}(s)\,g^{q-1}(s)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{\frac{2}{p}}(x)\,g^{\frac{2}{q}}(x)\right). \quad (44)$$

Multiplying both sides of Eq.(44), by $\frac{\rho^{1-\beta}x^{\kappa}s^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ and integrating with respect to variable $s$ on $(0,x)$, $x>0$, we conclude that

$$\begin{aligned}&\frac{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{q}(x)\right)}{p}+\frac{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p}(x)\right)}{q}\\ \geq\ &\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{\frac{2}{p}}(x)\,g^{\frac{2}{q}}(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{p-1}(x)\,g^{q-1}(x)\right).\end{aligned}$$

3. To prove item (3), we consider $a=\frac{f^{\frac{2}{p}}(t)}{g(s)}$ and $b=\frac{f^{\frac{2}{q}}(s)}{g(t)}$ with $g(s),g(t)\neq 0$ and substitute in the Young's inequality; the proof follows as in item (2).

□

**Theorem 5.** *Let $\alpha>0$, $\beta,\rho,\eta,\kappa\in\mathbb{R}$, $f,g\in X_{c}^{p}(0,x)$ two positive functions on $[0,\infty)$, $x>0$, and $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$. Let*

$$m=\min_{0\leq t\leq x}\frac{f(t)}{g(t)}\qquad\text{and}\qquad M=\max_{0\leq t\leq x}\frac{f(t)}{g(t)}. \quad (45)$$

*Then, the following inequalities follow:*

1. $0\leq\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}(x)\,{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}(x)\right)\leq\frac{(M+m)^{2}}{4Mm}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}(fg)(x)\right)^{2}$

2. $0\leq\sqrt{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}(x)\,{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}(x)}-\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}(fg)(x)\right)\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{2\sqrt{Mm}}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}(fg)(x)\right)$

3. $0\leq{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}(x)\,{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}(x)-\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}(fg)(x)\right)^{2}\leq\frac{(M-m)^{2}}{4Mm}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}(fg)(x)\right)^{2}$

**Proof.**

1. From Eq.(45) and the inequality

$$\left(\frac{f(t)}{g(t)}-m\right)\left(M-\frac{f(t)}{g(t)}\right)g^{2}(t)\geq 0,\ 0\leq t\leq x, \quad (46)$$

which we can write as

$$(f(t)-mg(t))\,(Mg(t)-f(t))\geq 0,$$

it follows that

$$(M+m)\,f(t)\,g(t)\geq f^{2}(t)+mMg^{2}(t). \quad (47)$$

Multiplying both sides of Eq.(47) by $\frac{\rho^{1-\beta}x^{\kappa}t^{\rho(\eta+1)-1}}{\Gamma(\alpha)(x^{\rho}-t^{\rho})^{1-\alpha}}$ and integrating with respect to variable $t$ on $(0,x)$, $x>0$, we find

$$(M+m)\,{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f(x)\,g(x)\geq\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}f^{2}(x)\right)+mM\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta}g^{2}(x)\right). \quad (48)$$

On the oher hand, it follows from $Mm > 0$ and

$$\left(\sqrt{{}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)} - \sqrt{mM\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right)}\right)^2 \geq 0, \tag{49}$$

that

$$\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right) + mM\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right) \geq 2\sqrt{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right)}\sqrt{mM\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right)}. \tag{50}$$

From Eq.(48) and Eq.(50) we get

$$2\sqrt{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right)}\sqrt{mM\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right)} \leq (M+m)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right). \tag{51}$$

Thus, we conclude that

$$\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right) \leq \frac{(M+m)^2}{4}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right)^2.$$

2. From Eq.(51) we have

$$\sqrt{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right)} \leq \frac{(M+m)}{2\sqrt{mM}}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right). \tag{52}$$

Subtracting ${}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)$ from both sides of Eq.(52), we get

$$\begin{aligned}\sqrt{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right)} - \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right) &\leq \frac{(M+m)}{2\sqrt{mM}}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right) \\ &\quad - \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right).\end{aligned}$$

Thus, we conclude that

$$\sqrt{\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f^2(x)\right)\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} g^2(x)\right)} - \left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right) \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}}\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right).$$

(3) Subtracting $\left({}^{\rho}\mathcal{I}_{\eta,\kappa}^{\alpha,\beta} f(x)\, g(x)\right)^2$ from both sides of Eq.(51) and following the same procedure used in item (2), we conclude the proof.

□

## Concluding remarks

From a fractional integral that unifies six different fractional integrals, as proposed by Katugampola, it was possible to generalize inequalities of Grüss type obtaining, as a particular case, the well-known inequality involving the Riemann-Liouville fractional integral. We also proved other inequalities using Katugampola's fractional integral. A natural continuation of this work would be the generalization of inequalities of Hermite-Hadamard and Hermite-Hadamard-Fejér types [23], as well as to propose inequalities using the so-called $M$-fractional integral recently introduced in [39].

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