COUNTING NUMERICAL SEMIGROUPS BY GENUS AND EVEN GAPS

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Abstract. We present an approach to count numerical semigroups of a given genus by using even gaps. Our method is motivated by the interplay between double covering of curves and γ-hyperelliptic semigroups [15], [12], [28], [18], [17].

1. Introduction

A *numerical semigroup* S is a submonoid of the set of nonnegative integers $\mathbb{N}_0$, equipped with the usual addition, such that $G(S) := \mathbb{N}_0 \setminus S$, the set of *gaps* of S, is finite. The number of elements $g = g(S)$ of $G(S)$ is called the *genus* of S and thus the semigroup property implies (see e.g. [13, Lemma2.14])

$$S \supseteq \{2g + i : i \in \mathbb{N}_0\} . \tag{1.1}$$

Suitable references for the background on numerical semigroups that we assume are in fact the books [13] and [22]. In spite of its simplicity, as a mathematical object, a numerical semigroup often plays a key role in the study of more involved or subtle structures arising e.g. in Algebraic Curve Theory [15], [12], [28], [18], [17] or e.g. in Coding Theory [21], [2].

In this paper we deal with a problem of purely combinatorial nature, namely: For $g \in \mathbb{N}_0$ given, find the number n_g of elements of the family $\mathcal{S}_g$ of numerical semigroups of genus g; one can find information on these numbers in Sloane's On-line Encyclopedia of Integer Sequences [25]. Indeed, our goal here is the question (1.2) below.

We have $n_g \leq \binom{2g-1}{g}$ by (1.1) and in fact, a better bound is known, namely $n_g \leq \frac{1}{g+1}\binom{2g}{g}$ which was obtained by Bras-Amorós and de Mier via so-called Dyck paths [7]. Further bounds on n_g were computed by Bras-Amorós [4] via the semigroup tree method; see also [6], [20], Elizalde [11]. Blanco and Rosales approached this problem by considering a partition of $\mathcal{S}_g$ by subsets of semigroups S of a given Frobenius number $F = F(S)$, which by definition is the biggest integer which does not belong to S; see also [1]. In any case, computing the exact value of n_g seems to be out of reach.

By taking into consideration the first 50 values of n_g, Bras-Amorós [5] conjectured Fibonacci-like properties on the behaviour of the sequence n_g:

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(A) $n_{g+2} \geq n_{g+1} + n_g$ for any g;
(B) $\lim_{g\to\infty} \frac{n_{g+1}+n_g}{n_{g+2}} = 1$;
(C) $\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \varphi := \frac{1+\sqrt{5}}{2}$, so-called *golden number*.

Indeed, Conjectures (B) and (C) have been recently proved by Zhai [29]. Here we focus in the following problem suggested by (A) whose answer is positive for large g by (C) or $g \leq 50$ by the aforementioned tables in [5]:

$$\text{It is true that } n_{g+1} > n_g \text{ for any } g \geq 1? \tag{1.2}$$

The *multiplicity* $m(S)$ of a numerical semigroup S is its first positive element. Kaplan [14] gave an approach to Conjecture A and Question (1.2) by counting numerical semigroups by genus and multiplicity. He obtained some partial interesting results, but his method does not solve the problems.

In addition, Bras-Amorós [3] introduced the notion of *ordinarization transform* $\mathbf{T} : \mathcal{S}_g \to \mathcal{S}_g$ given by $\mathbf{T}(S) = (S \cup \{F(S)\}) \setminus \{m(S)\}$, with $S \neq S_g := \{0\} \cup \{g+i : i \in \mathbb{N}\}$ (so called ordinary semigroup of genus g). Then the minimum nonnegative integer r such that $\mathbf{T}^r(S) = S_g$ is the *ordinarization number* of S; it turns out that $r \leq g/2$, and so she counted numerical semigroups by genus and ordinarization number. Unfortunately this method also does not give a positive answer to either computing n_g or question (1.2).

In this paper we approach (1.2) by counting numerical semigroups by genus and number of even gaps. Let $N_\gamma(g)$ denote the number of elements of the family $\mathcal{S}_\gamma(g)$, so-called *γ-hyperelliptic semigroups of genus g*; i.e. those in $\mathcal{S}_g$ whose number of even gaps equals γ. We have

$$n_g = \sum_{\gamma=0}^{\lfloor 2g/3 \rfloor} N_\gamma(g)\,, \tag{1.3}$$

by Corollary 2.3. In particular, (1.2) holds true provided that

$$N_i(g+1) > N_i(g) \quad \text{for } i = \lfloor g/3 \rfloor + 1, \ldots, \lfloor 2g/3 \rfloor\,. \tag{1.4}$$

In Section 2 we deal with the set of even gaps of a numerical semigroup, where the key result is Lemma 2.2 (cf. [27]). In particular, (1.3) is closely related to the stratification in (2.2). For $2g \geq 3\gamma$ (cf. [26]) we point out a quite useful parametrization, namely $\mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma$, $S \mapsto S/2$, which was introduced by Rosales et al. [23] (see (2.3)). Thus Remark 2.11 shows the class of numerical semigroups we deal with in this paper; we do observe that these semigroups were already studied for example in [19] by using the concept of *weight* of semigroups.

We have $N_\gamma(g) = N_\gamma(3\gamma)$ for $g \geq 3\gamma$, and $N_\gamma(g) < N_\gamma(3\gamma)$ for $g < 3\gamma$; see Corollary 3.5. The key ingredient here is the t-translation of a numerical semigroup introduced in Definition 3.1.

By the above considerations on $N_\gamma(g)$, it is natural to investigate the asymptotic behaviour of the sequence $N_\gamma(3\gamma)$ which is carry on in Section 4; indeed, to our surprise, it coincides with the sequence f_γ introduced by Bras-Amorós in [3, p. 2515]; see Theorem 4.4 here.

Finally in Section 5 we compute certain limits involving f_γ (see Proposition 5.1) which are of theoretical interest as they are related to the stronger possibility: $f_\gamma \sim \varphi^{2\gamma}$.

2. On the even gaps of a numerical semigroup

Throughout, let S be a numerical semigroup of genus $g = g(S)$, $G_2 = G_2(S)$ the set of its even gaps, and $\gamma = \gamma(S)$ the cardinality of G_2. As a matter of terminology, we say that S is *γ-hyperelliptic*. In particular, from (1.1), there are exactly $g-\gamma$ (resp. γ) even (resp. odd) nongaps in $S \cap [1, 2g]$. For $\gamma \geq 1$, these odd nongaps will be denoted by

$$o_\gamma = o_\gamma(S) < \ldots < o_1 = o_1(S)\,. \tag{2.1}$$

Notice that $o_i \leq 2g - 2i + 1$ for each i. As usual, for pairwise different natural numbers $a_1, \ldots, a_\alpha$, we set $\langle a_1, \ldots, a_\alpha\rangle := \{a_1x_1 + \ldots + a_\alpha x_\alpha : x_1, \ldots, x_\alpha \in \mathbb{N}_0\}$. It is well-known, so far, that this set is a numerical semigroup if and only if $\gcd(a_1, \ldots, a_\alpha) = 1$.

Remark 2.1. We have $\gamma(S) = 0$ if and only $S = \langle 2, 2g+1\rangle$; in the literature, this semigroup is classically called *hyperelliptic*. In general $g \geq \gamma$, and equality holds if and only if $g = \gamma = 0$.

From now on, we always assume $\gamma \geq 1$ so that $1, 2 \in G(S)$, the set of gaps of S, and $g \geq \gamma + 1$.

The following result and their corollaries were already noticed in [27]. It is analogous to (1.1), and for the sake of completeness we state proofs.

Lemma 2.2. *The biggest even gap ℓ of a γ-hyperelliptic semigroup S of genus g satisfies*

$$\ell \leq \min(4\gamma - 2, 4g - 4\gamma)\,.$$

Proof. Suppose that $\ell \geq 4\gamma$. Then in the interval $[2, 4\gamma-2]$ there are at least γ even nongaps of S says, $h_1 < \ldots < h_\gamma$. Thus S would have at least $\gamma+1$ even gaps, namely $\ell - h_\gamma < \ldots < \ell - h_1 < \ell$, a contradiction.

Now if $4\gamma - 2 \leq 4g - 4\gamma$; i.e., $g \geq 2\gamma$, the proof follows. Otherwise, consider $I := 2\gamma - g + 1$ which is a positive integer with $I \leq \gamma$ as $g \geq \gamma + 1$. Suppose that $\ell > o_I$, being (o_j) the sequence of odd nongaps of S in (2.1). Thus we obtain $\gamma - I + 1 = g - \gamma$ odd gaps of S, namely

$$\ell - o_I < \ldots < \ell - o_\gamma\,,$$

and hence $\ell = o_I + 1 \leq 2g - 2I + 2 = 4g - 4\gamma$. □

Corollary 2.3. (cf. [26]) *Let S be a γ-hyperelliptic semigroup of genus g. Then $2g \geq 3\gamma$.*

Proof. If $g \geq 2\gamma$, the result is clear. Let $g \leq 2\gamma - 1$. By Lemma 2.2, G_2 is contained in the interval $[2, 4g - 4\gamma]$ and hence $2g - 2\gamma \geq \gamma$ and we are done. □

Corollary 2.4. *Let S be a γ-hyperelliptic semigroup. Then its smallest odd nongap $O := o_\gamma(S)$ satisfies $O \geq \max(|2g - 4\gamma| + 1, 3)$.*

Proof. Clearly $O \geq 3$ since $\gamma \geq 1$. Let $g \geq 2\gamma$. By Lemma 2.2 in $[2, 4\gamma] \cap S$ there are exactly γ even nongaps, say $h_1 < \ldots < h_\gamma = 4\gamma$. Thus the elements O and $O + h_j$ are $\gamma + 1$ odd nongaps of S. Since S has exactly γ odd nongaps in $[1, 2g - 1]$, then $O + 4\gamma \geq 2g + 1$ and the result follows.

Now let $g \leq 2\gamma - 1$. Here, by Lemma 2.2, in $[2, 4g - 4\gamma] \cap S$ there are exactly $2g - 3\gamma$ even nongaps. Consider the sequence $2o_\gamma < \ldots < o_\gamma + o_{4\gamma - 2g}$ of $2g - 3\gamma + 1$ elements. Thus $o_\gamma + o_{4\gamma-2g} \geq 4g - 4\gamma + 2$. Since $o_{4\gamma-2g} \leq 6g - 8\gamma + 1$, we are done. □

As a way of illustration, next we describe 1-hyperelliptic and 2-hyperelliptic semigroups.

Example 2.5. Let $\gamma = 1$ and thus $g \geq 2$. Then $G_2 = \{2\}$ by Lemma 2.2 and $o_1 \geq \max(2g - 3, 3)$ by Corollary 2.4. Thus we obtain two types of 1-hyperelliptic semigroups of genus g, namely $S_1(g) = \langle 4, 6, 2g-3 \rangle$ with $g \geq 3$, and $S_2(g) = \langle 4, 6, 2g-1, 2g+1 \rangle$ with $g \geq 2$.

Example 2.6. Let $\gamma = 2$ and hence $g \geq 3$. Let $g = 3$. Then $G_2 = \{2, 4\}$ by Lemma 2.2 and thus $S = \langle 3, 5, 7 \rangle$. Now let $g \geq 4$. By Lemma 2.2 there is missing just one even nongap in $S \cap [4, 6]$, and by Corollary 2.4 $o_2 \geq \max(2g - 7, 3)$. For $g = 4$ we have the following four possibilities of 2-hyperelliptic semigroups: $\langle 3, 5 \rangle$, $\langle 3, 7, 8 \rangle$, $\langle 4, 5, 7 \rangle$, $\langle 5, 6, 7, 8, 9 \rangle$.

So let $g \geq 5$ and thus $o_2 \geq 2g - 7$. Here we obtain the following seven families of 2-hyperelliptic semigroups of genus g:

(1) $S_1(g) = \langle 4, 10, 2g - 7 \rangle$ with $g \geq 6$;
(2) $S_2(g) = \langle 4, 10, 2g - 5, 2g + 1 \rangle$;
(3) $S_3(g) = \langle 4, 10, 2g - 3, 2g - 1 \rangle$;
(4) $S_4(g) = \langle 6, 8, 10, 2g - 7 \rangle$;
(5) $S_5(g) = \langle 6, 8, 10, 2g - 5, 2g - 3 \rangle$;
(6) $S_6(g) = \langle 6, 8, 10, 2g - 5, 2g - 1 \rangle$;
(7) $S_7(g) = \langle 6, 8, 10, 2g - 3, 2g - 1, 2g + 1 \rangle$.

Remark 2.7. The examples above were already handled, among others, by Garcia [12] and Oliveira-Pimentel [18] who moreover noticed that all of them are Weierstrass semigroups; this property is also true for 3-hyperelliptic curves (see Komeda [17]). We point out that there are numerical semigroups which are not Weierstrass; cf. [28].

The following computations have to do with Corollary 2.3.

Example 2.8. (cf. [26]) We look for γ-hyperliptic semigroups S of genus g such that $g = \lceil 3\gamma/2 \rceil$.

Case γ even. For example for $\gamma = 2$ and $g = 3$, $S = \langle 3, 5, 7 \rangle$, as one can easily see from Example 2.6. In general, we show that S is generated by the set $\Sigma := \{\gamma + 2i - 1 : i = 1, \ldots, \gamma + 1\}$. Indeed, here $o_\gamma \geq \gamma + 1$ by Corollary 2.4. Since in $[\gamma + 1, 2g - 1]$ there are exactly γ odd numbers, then the $g - \gamma = \gamma/2$ odd gaps of S are precisely the odd numbers in $[1, \gamma - 1]$. On the other hand, Lemma 2.2 implies $G_2(S) \subseteq [2, 2\gamma]$ so that the $g - \gamma$ even numbers in $[2\gamma + 2, 2g]$ are even nongaps. Thus $G(S) = \{2i : i = 1, \ldots, \gamma\} \cup \{2i - 1 : i = 1, \ldots, \gamma/2\}$, or equivalently, S is generated by Σ as follows from e.g. [24, Sect. 3(III)].

Case γ odd. Here $2g = 3\gamma + 1$, $4g - 4\gamma = 2\gamma + 2$. If $\gamma = 1$ and hence $g = 2$, Example 2.5 shows that $S = \langle 3, 4, 5 \rangle$. Let $\gamma \geq 3$ and so Lemma 2.2 implies $G_2 \subseteq [2, 2\gamma + 2]$. Since in $[2\gamma + 4, 3\gamma + 1]$ we have $(\gamma - 1)/2 = g - \gamma - 1$ even numbers, S has just one even nongap x missing in the interval $[\gamma + 3, 2\gamma + 2]$. This gives $(\gamma + 1)/2$ possibilities for the selection of x $(*)$ so that the even nongaps in $S \cap [2, 2g]$ are the elements $\{x\} \cup \{2\gamma + 2 + 2i : i = 1, \ldots, (\gamma - 1)/2\}$.

Next we look for the odd nongaps of S; we have that $\gamma \leq o_\gamma \leq \gamma + 2$ by Corollary 2.4 and the definition of o_γ.

1. Let $o_\gamma = \gamma + 2$. In the interval $[\gamma + 2, 3\gamma]$ there are precisely γ odd integers and thus the set of odd nongaps of S in $S \cap [1, 2g]$ is $\{\gamma + 2i : i = 1, \ldots, \gamma\}$.

Then for each γ odd we obtain $(\gamma+1)/2$ γ-hyperelliptic semigroups of genus $g = (3\gamma+1)/2$.

2. Let $o_\gamma = \gamma$. In this case $x = 2\gamma$ in $(*)$ above. In the interval $[\gamma + 2, 3\gamma]$ there are γ odd numbers from which we have to choose $\gamma - 1$ of them. If $\gamma + 2 \in S$, $2\gamma + 2 \in S$, a contradiction. Thus the odd nongaps in $S \cap [1, 2g]$ are determined, namely those in the set $\{\gamma\} \cup \{\gamma + 3 + 2i - 1 : i = 1, \ldots, \gamma - 1\}$; i.e. we just obtain one numerical semigroup in this case.

Now we study a natural stratification of the family $\mathcal{S}_g$ defined above, by taking into consideration even gaps. As a matter of fact, we collect the subfamily of γ-hyperelliptic semigroups of genus g:

$$\mathcal{S}_\gamma(g) := \{S \in \mathcal{S}_g : \gamma(S) = \gamma\}, \quad \text{and thus}$$

$$\mathcal{S}_g = \bigcup_{\gamma=0}^{\lfloor 2g/3 \rfloor} S_\gamma(g) \tag{2.2}$$

by Corollary 2.3. The following definition was introduced by Rosales et al. [23] in connection with Diophantine inequalities; see also [27, p. 371], or the proof of [28, Scholium 3.5], where this concept is related to Stöhr's examples concerning symmetric semigroups which are not Weierstrass semigroups.

Definition 2.9. The *one half* of a numerical semigroup S is $S/2 := \{s \in \mathbb{N}_0 : 2s \in S\}$.

Another application of Lemma 2.2 is the following.

Corollary 2.10. *For a numerical semigroup S, $g(S/2) = \gamma(S)$.*

In particular, we have a natural parametrization of the family $\mathcal{S}_\gamma(g)$ onto $\mathcal{S}_\gamma$, where $2g \geq 3\gamma$, by means of the function

$$\mathbf{x} = \mathbf{x}_\gamma(g) : \mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma\,, \quad S \mapsto S/2\,. \tag{2.3}$$

This map is certainly surjective: Let $T \in \mathcal{S}_\gamma$, then $\mathbf{x}(S) = T$, where

$$S := 2T \cup \{2g - 2\gamma + i : i \in \mathbb{N}\} \in \mathcal{S}_\gamma(g)\,,$$

being $2T := \{2t : t \in T\}$; see also [8].

Remark 2.11. Indeed, any $S \in \mathcal{S}_\gamma(g)$ can be uniquely written as being:

$$S = 2(S/2) \cup \{o_\gamma < \ldots < o_1\} \cup \{2g + i : i \in \mathbb{N}_0\}\,,$$

where $o_\gamma, \ldots, o_1$ are certain odd numbers in $[O, 2g-1]$ with $O = \max\{|2g - 4\gamma| + 1, 3\}$ by Corollary 2.4.

3. On the family $\mathcal{S}_\gamma(g)$

In this section we deal with the family $\mathcal{S}_\gamma(g)$ of numerical semigroups of genus g whose number of even gaps equals γ. Throughout we assume $2g \geq 3\gamma$ (cf. Corollary 2.3).

Definition 3.1. Let $t \in \mathbb{Z}$. The t-translation of a numerical semigroup S is the map $\Phi_t : S \to \mathbb{Z}$ defined by

$$s \mapsto \begin{cases} s & \text{if } s \equiv 0 \pmod 2\,, \\ s - t & \text{otherwise}\,. \end{cases}$$

Lemma 3.2. *Let $t = 2g - 6\gamma$, $S \in \mathcal{S}_\gamma(g)$. Then $\Phi_t(S) \in \mathcal{S}_\gamma(3\gamma)$.*

Proof. We first show that $\Phi(S) := \Phi_t(S)$ is indeed a numerical semigroup. By Lemma 2.2 it is enough to notice that $2(o_\gamma(S) - t) \geq 4\gamma + 2$ which is clear from Corollary 2.4 and the selection of t. In particular, $\gamma(\Phi(S)) = \gamma$. Next we show that $g(\Phi(S)) = 3\gamma$.

Let $x = 2g + i \in S$, $i \in \mathbb{N}_0$. Then $x - t = 6\gamma + i \in \Phi(S)$ and the conductor c of this semigroup satisfies $c \leq 6\gamma$. In $[1, 6\gamma - 1]$ we have $(3\gamma - \gamma) = 2\gamma$ odd gaps of $\Phi_t(S)$, so that $g(\Phi(S)) = 3\gamma$. □

Thus Definition 3.1 with $t = 2g - 6\gamma$ induces a map

$$\tilde{\Phi}_t : \mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma(3\gamma)\,, \quad S \mapsto \Phi_t(S)\,.$$

Theorem 3.3. *The map $\tilde{\Phi}_t$ above is injective; it is bijective whenever $g \geq 3\gamma$.*

Proof. The map $\tilde{\Phi}_t$ is injective by its definition. Now for $T \in \mathcal{S}_\gamma(3\gamma)$, let us consider the $(-t)$-translation $\Phi_{(-t)} : T \to \mathbb{Z}$. Here we have $2(o_\gamma(T)+t) \geq 2(2\gamma+1+2g-6\gamma) \geq 4\gamma+2$ by Corollary 2.4 and $g \geq 3\gamma$. Thus we have a map $\tilde{\Phi}_{(-t)} : \mathcal{S}_\gamma(3\gamma) \to \mathcal{S}_\gamma(g)$ induced by $\Phi_{(-t)}$ which is clearly the inverse of $\tilde{\Phi}_t$. □

Remark 3.4. Let $g < 3\gamma$ so $t = 2g-6\gamma \leq -2$. Then the map $\tilde{\Phi}_t$ above is not surjective. Indeed, let $S := \langle 4, 2\gamma+1 \rangle$ which belongs to $\mathcal{S}_\gamma(3\gamma)$. Suppose there exists $T \in \mathcal{S}_\gamma(g)$ such that $\tilde{\Phi}_t(T) = S$. Then $o_\gamma(T) = 2\gamma+1+t$ so that $h := 2o_\gamma(T) = 4\gamma+2+2t \in T$ with $h \in \Sigma := \{\ell \in \mathbb{N} : \ell \equiv 2 \pmod 4, 2 \leq \ell \leq 4\gamma-2\}$. It turns out that $\Sigma \subseteq G(T)$, a contradiction.

Recall that $N_\gamma(g) = \#\mathcal{S}_\gamma(g)$.

Corollary 3.5. *$N_\gamma(g) \leq N_\gamma(3\gamma)$; equality holds whenever, $g \geq 3\gamma$, and $N_\gamma(g) < N_\gamma(3\gamma)$ otherwise.*

Next we show a table for some values of $N_\gamma(g)$; we obtain such computations by using the GAP package [9].

g \ γ	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	2	1							
4	1	2	4							
5	1	2	6	3						
6	1	2	7	12	1					
7	1	2	7	19	10					
8	1	2	7	21	32	4				
9	1	2	7	23	51	33	1			
10	1	2	7	23	62	91	18			
11	1	2	7	23	65	142	98	5		
12	1	2	7	23	68	174	257	59	1	
13	1	2	7	23	68	192	412	271	25	
14	1	2	7	23	68	197	514	678	197	6
15	1	2	7	23	68	200	570	1100	793	92
16	1	2	7	23	68	200	602	1409	1855	606
17	1	2	7	23	68	200	609	1595	2999	2191
18	1	2	7	23	68	200	615	1693	3890	4993
19	1	2	7	23	68	200	615	1744	4472	8126
20	1	2	7	23	68	200	615	1756	4797	10723
21	1	2	7	23	68	200	615	1764	4959	12528
22	1	2	7	23	68	200	615	1764	5034	13616
23	1	2	7	23	68	200	615	1764	5053	14191
24	1	2	7	23	68	200	615	1764	5060	14469
25	1	2	7	23	68	200	615	1764	5060	14589
26	1	2	7	23	68	200	615	1764	5060	14611
27	1	2	7	23	68	200	615	1764	5060	14626

TABLE 1. A few values for $N_\gamma(g)$

$g \backslash \gamma$	10	11	12	13	14	15	16	17	18	n_g
0										1
1										1
2										2
3										4
4										7
5										12
6										23
7										39
8										67
9										118
10										204
11										343
12										592
13										1001
14										1693
15	1									2857
16	33									4806
17	343	7								8045
18	1836	138	1							13467
19	6033	1130	43							22464
20	13317	5335	544	8						37396
21	21764	16447	3624	191	1					62194
22	29209	35392	15365	1897	53					103246
23	34628	57925	44575	11098	804	9				170963
24	38096	78602	93919	43262	6485	254	1			282828
25	40098	94469	154077	119669	33525	3013	64			467224
26	41086	105074	211576	247756	120881	20945	1153	10		770832
27	41541	111426	257734	407238	320649	98104	10873	335	1	1270267

TABLE 2. A few values for $N_\gamma(g)$ (cont.)

These computations suggest a positive answer to both questions (1.4) and (1.2) above.

We end up this section by pointing out a result concerning specific properties of semigroups S in the fiber $\mathbf{x}^{-1}(T)$ in (2.3), where $T \in \mathcal{S}_\gamma$. For example, for $g, \gamma \in \mathbb{N}_0$ with $g \geq 3\gamma$, let us consider Stöhr's examples in [28, p. 48]:

$$S := 2T \cup \{2g - 1 - 2t : t \in \mathbb{Z} \setminus T\}$$

which are γ-hyperelliptic symmetric semigroups of genus g. Thus we have:

Scholium 3.6. Let g and γ be integers such that $g \geq 3\gamma$. Then there exists, at least, n_γ γ-hyperelliptic symmetric semigroups of genus g.

4. On the sequence f_γ

This section is closely related to Bras-Amorós approach [3]; see Theorem 4.4.

Definition 4.1. Let S be a numerical semigroup.

(1) A set $B \subseteq \mathbb{N}_0$ is called *S-closed* if for $b \in B$, $s \in S$ we have either $b+s \in B$, or $b+s > \max(B)$.
(2) We let $C(S,i)$ denote the collection of S-closed sets B such that $0 \in B$ and $\#B = i$.

Lemma 4.2. *Let* $S \in \mathcal{S}_\gamma$, $B \in C(S, \gamma+1)$. *Then* $\max(B) \leq 2\gamma$.

Proof. Suppose $\max(B) > 2\gamma$. Then $F := \#[0,2\gamma] \cap S \leq \gamma$; since $g(S) = \gamma$, $F = \gamma+1$ which gives rise to a contradiction. □

Definition 4.3. ([3]) For $\gamma \in \mathbb{N}_0$, $f_\gamma := \sum_{S \in \mathcal{S}_\gamma} \#C(S, \gamma+1)$.

The main result of this section is the following. Notation as in Section 3.

Theorem 4.4. *For* $\gamma \in \mathbb{N}_0$, $f_\gamma = N_\gamma(3\gamma) = \#\mathcal{S}_\gamma(3\gamma)$.

Proof. Let $\mathbf{x} = \mathbf{x}_\gamma(3\gamma) : \mathcal{S}_\gamma(3\gamma) \to \mathcal{S}_\gamma$, $S \mapsto S/2$ (see Definition 2.9). The result follows from the following computations.

Claim. There is a bijective map $\mathbf{F}$ between the sets $C(T, \gamma+1)$ and $\mathbf{x}^{-1}(T)$ with $T \in \mathcal{S}_\gamma$.

In fact, for $B \in C(T, \gamma+1)$ we let $\mathbf{F}(B) = 2T \cup \{2b - 2\max(B) + 6\gamma + 1 : b \in B\}$; this map is well defined by Lemma 4.2.

Now let $S \in \mathbf{x}^{-1}(T)$ so that $S = 2T \cup \{o_\gamma(S) < \ldots < o_1(S) < o_0 := 6\gamma+1\} \cup \{6\gamma + i : i \in \mathbb{N}_0\}$ with o_i odd integers. Set $o_i(S) = o_i$ and define $b_i := (o_i - o_\gamma)/2$, $i = 0, \ldots, \gamma$. By definition it is clear that $B := \{b_0, \ldots, b_\gamma\} \in C(T, \gamma+1)$ and the inverse map of $\mathbf{F}$ is given by $S \mapsto B$. □

Next we investigate bounds on the sequence f_γ by taking advantage of Theorem 4.4 above; thus we shall be dealing with sets of the form:

$$S = 2T \cup \mathcal{O} \cup \{6\gamma + j : j \in \mathbb{N}_0\}, \tag{4.1}$$

where $T \in \mathcal{S}_\gamma$, and $\mathcal{O} = \{o_\gamma < \ldots < o_1\}$ is certain set of γ odd integers in $[2\gamma+1, 6\gamma-1]$.

Remark 4.5. The set S in (4.1) belongs to $\mathcal{S}_\gamma(3\gamma)$ if and only if for $t \in T$, $o_j \in \mathcal{O}$ we have $2t + o_j \in \mathcal{O}$ or $2t + o_j > 6\gamma$.

Throughout, we let

$$o_\gamma = 2\gamma + 2i + 1 \quad \text{for some } i \in \{0, \ldots, \gamma\}\,.$$

In addition we set:

$$\mathbf{x}^{-1}(T^i) := \{S \in \mathcal{S}_\gamma(3\gamma) : S/2 = T,\, o_\gamma(S) = 2\gamma + 2i + 1\}\,, \tag{4.2}$$

where $\mathbf{x} = \mathbf{x}_\gamma(g)$ is the map in (2.3) with $g = 3\gamma$. We notice that $\mathbf{x}^{-1}(T) = \cup_{i=0}^{\gamma}\mathbf{x}^{-1}(T^i)$, and $\mathcal{S}_\gamma(3\gamma) = \cup_{T\in\mathcal{S}_\gamma}\mathbf{x}^{-1}(T)$.

Lemma 4.6. *Let $T \in \mathcal{S}_\gamma$. With the above notation,*

$$1 \le \#\mathbf{x}^{-1}(T^i) \le \binom{\gamma}{i}\,.$$

Proof. Let $t_0 = 0 < t_1 < \ldots <$ be the enumeration of T in increasing order. By (1.1) $t_{\gamma+j} = 2\gamma + j$ for any $j \in \mathbb{N}_0$. Set

$$\mathcal{O}(1) = \{o_\gamma + 2t : t \in T,\, t \le 2\gamma - i - 1\}\,.$$

In (4.1) let us write $\mathcal{O}$ as the disjoint union of $\mathcal{O}(1)$ and certain set $\mathcal{O}(2)$. If the elements of $\mathcal{O}(2)$ are the largest odd integers in

$$[2\gamma + 1, 6\gamma - 1] \setminus \mathcal{O}(1)\,,$$

then S in (4.1) belongs to $S_\gamma(3\gamma)$ and $\#\mathbf{x}^{-1}(T^i) \ge 1$. Since $t_{\gamma-i-1} \le 2\gamma - i - 1$, so $\#\mathcal{L}(2) \le i$ and the upper bound follows. □

Remark 4.7. The set $\mathcal{O}$ for both the extreme cases $i = 0, \gamma$ in Lemma 4.6 is easy to describe. In fact here we have $\mathcal{O} = \{2\gamma + 1 + 2t_j : j = 0, \ldots, \gamma\}$ (resp. $\mathcal{O} = \{4\gamma + 2j - 1 : j = 1, \ldots \gamma\}$) for $i = 0$ (resp. $i = \gamma$). Thus

$$\#\mathbf{x}^{-1}(T^0) = \#\mathbf{x}^{-1}(T^\gamma) = 1\,.$$

From now on, unless otherwise stated, we consider $1 \le i \le \gamma - 1$.

Corollary 4.8. *Let $\gamma \in \mathbb{N}_0$. Then*

$$n_\gamma \cdot (\gamma + 1) \le f_\gamma \le n_\gamma \cdot 2^\gamma\,.$$

Proof. It follows from Theorem 4.4, Lemma 4.6 and the well-known fact $\sum_{i=0}^{\gamma}\binom{\gamma}{i} = 2^\gamma$. □

Remark 4.9. From Corollaries 3.5 and 4.8,

$$N_\gamma(g) \le N_\gamma(3\gamma) = f_\gamma \le n_\gamma \cdot 2^\gamma\,.$$

Corollary 4.10. *Let $\varphi = (\sqrt{5} + 1)/2$ be the golden ratio.*

(1) *For $\epsilon > 0$, $\lim_{\gamma\to\infty} \frac{f_\gamma}{(2\varphi+\epsilon)^\gamma} = 0$;*
(2) *We have $\lim_{\gamma\to\infty} \frac{f_\gamma}{\varphi^\gamma} = \infty$.*

Proof. For $\gamma \in \mathbb{N}_0$, let $A_\gamma := \frac{n_\gamma}{\varphi^\gamma}$, $B_\gamma := \left(\frac{2\varphi}{2\varphi+\epsilon}\right)^\gamma$. We notice that $\lim_{\gamma\to\infty} A_\gamma$ is a real number by [29, Thm. 1].

(1) By Corollary 4.8 $f_\gamma/(2\varphi+\epsilon)^\gamma \leq A_\gamma \cdot B_\gamma$. Since $\lim_{\gamma\to\infty} B_\gamma = 0$, the proof follows.

(2) From Corollary 4.8 $f_\gamma/\varphi^\gamma \geq A_\gamma(\gamma+1)$, and we are done. □

In the remainder part of this section we shall be dealing with Remark 4.5 toward an improvement of Corollary 4.8 (see Corollary 4.16 below). We start by spliting off the set of odd integers in the interval $[2\gamma+1, 6\gamma-1]$ into $\mathcal{O}$ and $\mathcal{L} = \{\omega_1 < \ldots < \omega_\gamma\}$. Recall that $o_\gamma = 2\gamma + 2i + 1$ for some $i \in \{1, \ldots, \gamma-1\}$. Then we have a disjoint union $\mathcal{L} = \mathcal{L}^i(1) \cup \mathcal{L}^i(2)$, where

$$\mathcal{L}^i(1) := \{2\gamma+1 < \ldots < 2\gamma+2i-1\}, \quad \text{and}$$

$$\mathcal{L}^i(2) := \{\omega_{i+1} < \ldots < \omega_\gamma\} \subseteq \{o_\gamma + 2q : q \in G(T), q \leq 2\gamma - i - 1\}.$$

Lemma 4.11. *Let $T \in \mathcal{S}_\gamma$, $q, \bar{q} \in G(T)$ such that $\bar{q} - q = t \in T$. If S in (4.1) is a numerical semigroup, then $o_\gamma + 2q \in \mathcal{L}^i(2)$, whenever $o_\gamma + 2\bar{q}$ does.*

Proof. We have $o_\gamma + 2\bar{q} = o_\gamma + 2q + 2t$ so that $o_\gamma + 2q \in \mathcal{L}^i(2)$. □

Let us work out a numerical example.

Example 4.12. For the numerical semigroup $T = \mathbb{N}_0 \setminus \{1, 2, 3, 6\}$ of genus $\gamma = 4$, we shall compute $\#\mathbf{x}^{-1}(T^i)$ for $i = 1, 2, 3$ so that that $\#\mathbf{x}^{-1}(T) = 10$ and hence

$$5n_4 + 5 \leq f_4 \leq 16n_4 - 6.$$

(1) If $i = 1$, $o_4 = 11$,

$$\mathcal{L}^1(2) = \{\omega_2 < \omega_3 < \omega_4\} \subseteq \{11 + 2q : q \in G(T), q \leq 6\} = \{13, 15, 17, 23\}.$$

Since $23 = 11 + 2 \times 6$ by Lemma 4.11 $\mathcal{L}^1(2)$ can be either $\{13, 15, 17\}$ or $\{13, 15, 23\}$; it is a matter of fact that these computations define semigroups S in (4.1) so that $\#\mathbf{x}^{-1}(T^1) = 2$.

(2) If $i = 2$, $o_4 = 13$,

$$\mathcal{L}^1(2) = \{\omega_3 < \omega_4\} \subseteq \{13 + 2q : q \in G(T), q \leq 5\} = \{15, 17, 19\},$$

so that $\#\mathbf{x}^{-1}(T^2) = \binom{3}{2} = 3$.

(3) If $i = 3$, $o_4 = 15$,

$$\mathcal{L}^1(2) = \{\omega_4\} \subseteq \{15 + 2q : q \in G(T), q \leq 4\} = \{17, 19, 21\},$$

so that $\#\mathbf{x}^{-1}(T^3) = \binom{3}{1} = 3$.

Next we generalize this example. Let $k \in \{0, \ldots, \gamma - 1\}$ and consider the set

$$T = T_k = \mathbb{N}_0 \setminus \{1, \ldots, \gamma - 1, \gamma + k\}$$

which is a numerical semigroup of genus γ by the selection of k. We shall compute $\#\mathbf{x}^{-1}(T^i)$ for $T = T_k$, $0 \leq i \leq \gamma$. With notation as above

$$\mathcal{L}^i(2) \subseteq \{o_\gamma + 2q : q \in G(T)\,, q \leq 2\gamma - i - 1\} =$$
$$\begin{cases} \{o_\gamma + 2q : 1 \leq q \leq \gamma - 1\} \cup \{o_\gamma + 2(\gamma + k)\}\,, & \text{if } i + k \leq \gamma - 1\,, \\ \{o_\gamma + 2q : 1 \leq q \leq \gamma - 1\}\,, & \text{if } i + k > \gamma - 1\,. \end{cases}$$

Lemma 4.13. *Notation as above. Let $i \in \{0, \ldots, \gamma\}$, $k \in \{0, \ldots, \gamma - 1\}$.*

(1) *For $T = T_0$, $\#\mathbf{x}^{-1}(T^i) = \binom{\gamma}{i}$;*
(2) *Let $k \geq 1$, $T = T_k$. If $i + k \leq \gamma - 1$, then*
$$\#\mathbf{x}^{-1}(T^i) = \binom{\gamma - k - 1}{i} + \binom{\gamma - 1}{i - 1}\,;$$
(3) *Let $k \geq 1$, $T = T_k$. If $i + k \geq \gamma$, then*
$$\#\mathbf{x}^{-1}(T^i) = \binom{\gamma - 1}{i - 1}\,.$$

Proof. (1) For $k = 0$ we have $\mathcal{L}^i(2) \subseteq \{o_\gamma + 2q : 1 \leq q \leq \gamma\}$. Thus the number of sets of type as in (4.1) equals $\binom{\gamma}{\gamma - i} = \gamma\gamma i$; all such sets belong to $\mathcal{S}_\gamma(3\gamma)$ by Remark 4.5; the result follows.

(2) Here $\mathcal{L}^i(2) \subseteq \{o_\gamma + 2q : 1 \leq q \leq \gamma - 1\} \cup \{o_\gamma + 2(\gamma + k)\}$. If $o_\gamma + 2(\gamma + k) \in \mathcal{L}^i(2)$, then $o_\gamma + 2q \in \mathcal{L}^i(2)$ by Lemma 4.11. Thus we obtain $\binom{\gamma - k - 1}{\gamma - i - k - 1} = \binom{\gamma - k - 1}{i}$ sets of type (4.1) which belong to $\mathcal{S}_\gamma(3\gamma)$ by Remark 4.5. On the other hand, if $o_\gamma \notin \mathcal{L}^i(2)$ arguing as above we obtain further $\binom{\gamma - 1}{\gamma - i} = \binom{\gamma - 1}{i - 1}$ numerical semigroups in $\mathcal{S}_\gamma(3\gamma)$.

(3) In this case $\mathcal{L}^i(2) \subseteq \{o_\gamma + 2q : 1 \leq \gamma - 1\}$ and arguing as in (1) $\#\mathbf{x}^{-1}(T^i) = \binom{\gamma - 1}{i - 1}$. $\square$

By summing up the computations in Lemma 4.13, we obtain:

Corollary 4.14. *Notation as above. For $k = 0, \ldots, \gamma - 1$,*
$$\#\mathbf{x}^{-1}(T_k) = 2^{\gamma - 1 - k}(2^k + 1)\,.$$

Remark 4.15. The weight of the semigroup $T = T_k$ is $w_k = k$. We have $w_k \leq \gamma/2$, or $\gamma/2 < w_k \leq \gamma - 1$ and $2\gamma > \gamma + k$; thus T is Weierstrass [10], [16]. Then the unique element in $\mathbf{x}^{-1}(T^\gamma)$ is also Weierstrass by [17, Prop. 2.4].

Set
$$M_\gamma := \sum_{k=0}^{\gamma - 1} \#\mathbf{x}^{-1}(T_k) = 2^{\gamma - 1}(\gamma + 2) - 1\,.$$
Then, after some computations, Corollary 4.8 can be improved as follows.

Corollary 4.16. *With notation as above,*

$$c_1(\gamma) \leq f_\gamma \leq c_2(\gamma)\,,$$

where $c_1(\gamma) := n_\gamma(\gamma+1) + M_\gamma - \gamma(\gamma+1)$, *and* $c_2(\gamma) := n_\gamma \cdot 2^\gamma - (\gamma \cdot 2^\gamma - M_\gamma)$.

Remark 4.17. From Corollary 4.16, [4] we obtain the following computations.

γ	$c_1(\gamma)$	$f_\gamma = N_\gamma(3\gamma)$	$c_2(\gamma)$
0	1	1	1
1	2	2	2
2	7	7	7
3	23	23	27
4	62	68	95
5	153	200	266
6	374	615	1343
7	831	1764	4671
8	1810	5060	16383
9	3905	14626	52993
10	8277	41785	192513
11	17295	117573	666625
12	36211	332475	2347009
13	75271	933891	8032257
14	156256	2609832	27377665

TABLE 3. Bounds for f_γ

In addition, we improve Corollary 4.10(2) above as follows.

Corollary 4.18. $\lim_{\gamma\to\infty}\frac{f_\gamma}{2^\gamma} = \infty$.

5. Further results on the sequence f_γ

From Corollaries 4.10, 4.14, and the column regarding $f_\gamma/f_{\gamma-1}$ in Table 3 below, it seems that the following property holds true:

$$\textbf{(C:)} \qquad f_\gamma \sim \varphi^{2\gamma}\ ,$$

where as usual φ is the golden ratio. We end up by computing some interesting limits involving the sequence f_γ and which are very much related to statement **(C)** above. Recall that $\lim_{g\to\infty}\frac{n_{g+1}}{n_g} = \varphi$ [29].

Proposition 5.1. (1) **(C)** *is equivalent to* $f_\gamma \sim n_{2\gamma}$;

(2) **(C)** *implies* $\lim_{\gamma\to\infty}\frac{f_{\gamma+1}}{f_\gamma} = \varphi^2$;

γ	f_γ	$n_{2\gamma}$	$f_\gamma/f_{\gamma-1}$	$f_\gamma/n_{2\gamma}$	$f_{\gamma+1}/\sum_{i=0}^{\gamma} f_i$
0	1	1		1.00	2.00
1	2	2	2.00	1.00	2.33
2	7	7	3.50	1.00	2.30
3	23	23	3.29	1.00	2.06
4	68	67	2.96	1.01	1.98
5	200	204	2.94	0.98	2.04
6	615	592	3.08	1.04	1.93
7	1764	1693	2.87	1.04	1.89
8	5060	4806	2.87	1.05	1.89
9	14626	13467	2.89	1.09	1.87
10	41785	37396	2.86	1.12	1.83
11	117573	103246	2.81	1.14	1.83
12	332475	282828	2.83	1.18	1.82
13	933891	770832	2.81	1.21	1.80
14	2609832	2091030	2.79	1.25	

TABLE 4

(3) *If* $\lim_{\gamma\to\infty}\frac{f_\gamma+1}{f_\gamma}=\varphi^2$, *then* $\lim_{\gamma\to\infty}\frac{f_{\gamma+1}}{\sum_{i=0}^{\gamma} f_i}=\varphi$.

Proof. (1) By [29, Thm. 1] $\varphi^{2\gamma}\sim n_{2\gamma}$, so the result follows.

(2) Write $\frac{f_{\gamma+1}}{f_\gamma}=\frac{f_{\gamma+1}}{n_{2\gamma+2}}\cdot\frac{n_{2\gamma+2}}{n_{2\gamma+1}}\cdot\frac{n_{2\gamma+1}}{n_{2\gamma}}\cdot\frac{n_{2\gamma}}{f_\gamma}$. By (1), $\lim_{\gamma\to\infty}\frac{f_\gamma}{n_{2\gamma}}=K>0$. Then

$$\lim_{\gamma\to\infty}\frac{f_{\gamma+1}}{f_\gamma}=K\cdot\varphi\cdot\varphi\cdot\frac{1}{K}=\varphi^2\,.$$

(3) Let $0<\epsilon<1/3$.

Claim.

$$M:=\frac{1-\epsilon\varphi^2}{\varphi^2-(1-\epsilon\varphi^2)}\leq\lim_{\gamma\to\infty}\frac{f_0+\ldots+f_\gamma}{f_{\gamma+1}}\leq F:=\frac{1+\epsilon\varphi^2}{\varphi^2-(1+\epsilon\varphi^2)}\,.$$

Then (3) follows after letting $\epsilon\to 0$ and from the well-known fact that $\varphi^2=\varphi+1$.

Proof of the Claim. By hypothesis, $\lim_{\gamma\to\infty} f_\gamma/f_{\gamma+1}=1/\varphi^2$. Set

$$\gamma_0(\epsilon):=\min\{i\in\mathbb{N}:\frac{1}{\varphi^2}-\epsilon<\frac{f_j}{f_{j+1}}<\frac{1}{\varphi^2}+\epsilon,\,\forall j\geq i\}\,.$$

For $\gamma>\gamma_0=\gamma_0(\epsilon)=\gamma_0$ write

$$\frac{f_0+\ldots+f_{\gamma_0-1}+f_{\gamma_0}+\ldots+f_\gamma}{f_{\gamma+1}}=\frac{f_0+\ldots+f_{\gamma_0-1}}{f_{\gamma+1}}+\sum_{j=\gamma_0}^{\gamma}A_j\,,$$

where $A_j = f_j/f_{\gamma+1}$. In particular,

$$A_j = \frac{f_j}{f_{j+1}} \cdot \ldots \cdot \frac{f_\gamma}{f_\gamma} < t^{\gamma-j+1},$$

being $t = \frac{1}{\varphi^2} + \epsilon < 1$ so that

$$\lim_{\gamma\to\infty} \sum_{j=\gamma_0}^{\gamma} A_j = F$$

and we obtain the upper bound. We can prove the lower bound in a similar way. □

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References

[1] V. Blanco, P.A. García-Sánchez and J. Puerto, *Computing numerical semigroups with short generating functions*, Int. J. Algebra Comput. **21** (2011), 1217–1235.

[2] M. Bras-Amorós, "Semigroups and codes", 167–218, Algebraic Geometry Modeling in Information Theory" E. Martinez-Moro (Ed.) Word Scientific, 2013.

[3] M. Bras-Amorós, *The ordinarization transform of a numerical semigroup and semigroups with a large number of intervals*, J. Pure Appl. Algebra **216** (2012), 2507–2518.

[4] M. Bras-Amorós, *Bounds on the number of numerical semigroups of a given genus*, J. Pure Appl. Algebra **213**(6) (2009), 997–1001.

[5] M. Bras-Amorós, *Fibonacci-like behavior of the number of numerical semigroups of a given genus*, Semigroup Forum **76** (2008), 379–384.

[6] M. Bras-Amorós and S. Bulygin, *Towards a better understanding of the semigroup tree*, Semigroup Forum **79** (2009), 561–574.

[7] M. Bras-Amorós and A. de Mier, *Representation of numerical semigroups by Dyck paths*, Semigroup Forum **75** (2007), 676–681.

[8] M. D'Anna and F. Strazzanti, *The numerical duplication of a numerical semigroup*, Semigroup Forum **87** (2013), 149–160.

[9] M. Delgado, P.A. García-Sánchez, and J. Morais, "NumericalSgps, A package for numerical semigroups", Version 1.0.1 (2015), http://www.fc.up.pt/cmup/mdelgado/numericalsgps

[10] D. Eisenbud and J. Harris, *Existence, decomposition and limits of certain Weierstrass points*, Invent. Math. **87** (1987), 495–515.

[11] S. Elizalde, *Improved bounds on the number of numerical semigroups of a given genus*, J. Pure Appl. Algebra **214** (2010), 1862–1873.

[12] A. Garcia, *Weights of Weierstrass points in double covering of curves of genus one or two*, Manuscripta Math. **55** (1986), 419–432.

[13] P.A. García-Sánchez and J.C. Rosales, "Numerical semigroups", Developments in Mathematics vol. **20**, Springer, New York, 2009.

[14] N. Kaplan, *Counting numerical semigroups by genus and some cases of a question of Wilf*, J. Pure Appl. Algebra **216** (2012), 1016–1032.

[15] T. Kato, *On criteria of $\tilde{g}$-hyperellipticity*, Kodai Math. J. **2** (1979), 275–285.

[16] J. Komeda, *On primitive Shubert indices of genus g and weight $g-1$*, J. Math. Soc. Japan **43**(3) (1991), 437–445.

[17] J. Komeda, *On Weierstrass semigroups of double coverings of genus three curves*, Semigroup Forum **83** (2011), 479–488.

[18] G. Oliveira and F.L.R. Pimentel, *On Weierstrass semigroups of double covering of genus two curves*, Semigroup Forum **77** (2008), 152–162.

[19] G. Oliveira, F. Torres, and J. Villanueva, *On the weight of numerical semigroups*, J. Pure Appl. Algebra **214** (2010), 1955–1961.

[20] E. O'Dorney, *Degree asymptotics of the numerical semigroup tree*, Semigroup Forum **87** (2013), 601–616.

[21] R. Pellikaan and F. Torres, *On Weierstrass semigroups and the redundancy of improved geometric Goppa codes*, IEEE Trans. Inform. Theory **45**(7) (1999), 2512-2519.

[22] J.L. Ramírez-Alfonsín, "The Diophantine Frobenius Problem", Oxford Univ. Press vol **30**, 2005.

[23] J.C. Rosales, P.A. García-Sánchez, J.I. García-Sánchez, J.M. Urbano-Blanco, *Proportionally modular Diophantine inequalities*, J. Number Theory **103** (2003), 281–294.

[24] E.S. Selmer, *On the linear Diophantine problem of Frobenius*, J. Reine Angew. Math. **293/294** (1977), 1–17.

[25] N.J.A. Sloane, "The On-Line Encyclopedia of Integer Sequences", A007323, http:// www.research.att.com/ njas/sequences/(2009)

[26] F. Strazzanti, *Minimal genus of a multiple and Frobenius number of a quotient of a numerical semigroup*, Int. J. Algebra Comput. **25** (2015), 1043–1053.

[27] F. Torres, *On γ-hyperelliptic numerical semigroups*, Semigroup Forum **55** (1997), 364–379.

[28] F. Torres, *Weierstrass points and double coverings of curves. With application: Symmetric numerical semigroups which cannot be realized as Weierstrass semigroups*, Manuscripta Math. **83** (1994), 39–58.

[29] A. Zhai, *Fibonacci-like growth of numerical semigroups of a given genus*, Semigroup Forum **86** (2013), 634–662.

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