NEAR WEIGHTS ON HIGHER DIMENSIONAL VARIETIES

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Abstract. We generalize the concept of near weight stated in [2007, *IEEE Trans. Inform. Theory* **53**(5), 1919–1924] in the sense that we consider maps to arbitrary well-ordered semigroups instead of the nonnegative integers. This concept can be used as a tool to study AG codes based on more than one point via elementary methods only.

1. Introduction

Algebraic-geometric codes (AG codes) are linear error-correcting codes constructed by Goppa, around 1977, from algebraic curves over finite fields [9], [10]. The dimension and minimum distance of these codes can be handled by using tools from Algebraic Geometry such as the Riemann-Roch theorem. The subject of AG codes become important soon in Coding Theory after Tsfasman, Vladut and Zink [19] showed that the Varshamov-Gilbert bound can be attained by using these codes.

A more elementary way of describing linear codes was introduced by Høholdt, van Lint and Pellikaan around 1998 [11] based on the concept of *order (weight) function*. This technique allows us to describe mainly one-point AG codes [13]. A generalization of this idea to two-point codes was given by Carvalho et al. [3], Silva [17] by means of the concept of *near order (weight) function* which is a natural generalization of a order (weight) function; see also [4]. Geil and Pellikaan [8] generalized the concept of order (weight) function in the sense that they considered maps to arbitrary well-ordered semigroups instead of the nonnegative integers $(\mathbb{N}_0, \leq)$; see also [7], [1].

In this paper we are mainly interested in generalizing the approach in [3] and [8] by means of Definition 3.1 below. In Section 2, in order to provide a framework for our results and for the convenience of the reader, we stay some facts on Valuation Theory. Our main reference is Vaquié's paper [20] (see also [2], [21]). In Sections 3.1, 3.2 we present some generalized near weight functions constructed from certain valuation rings and toric rings. In Section 4 we introduce the concept of admissible near weight structures and we investigate certain linear codes. Finally, in Section 5 we characterize $\mathbb{F}$-algebras admitting two well-agreeing near weights where Theorem 5.13 generalizes [14, Thm. 5.6].

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2. Background on Valuations

Let $\Gamma = (\Gamma, +, \preceq)$ be an (additive, commutative) group equipped with an admissible total order $\preceq$. For a symbol $\infty \notin \Gamma$, let $\Gamma_\infty := \Gamma \cup \{\infty\}$; we extend $\preceq$ to this set by declaring $\alpha \preceq \infty$, $\alpha + \infty = \infty + \alpha = \infty$ for every $\alpha \in \Gamma_\infty$. Let $\mathbf{K}$ be a field. A map $\nu : \mathbf{K} \to \Gamma_\infty$ is called a *valuation* of $\mathbf{K}$ with *value group* Γ if the following conditions are satisfied:

(1) $\nu(x) = \infty \quad$ if and only if $\quad x = 0$;
(1) $\nu(xy) = \nu(x) + \nu(y)$;
(2) $\nu(x + y) \succeq \min\{\nu(x), \nu(y)\}$.

For a valuation ν of $\mathbf{K}$ with value group Γ, the *valuation ring* (resp. *maximal ideal*) associated to ν is the local ring (resp. maximal ideal) $\mathbf{R}_\nu := \{f \in \mathbf{K} : 0 \preceq \nu(f)\}$ (resp. $\mathbf{M}_\nu = \{f \in \mathbf{K} : 0 \preceq \nu(f), \nu(f) \neq 0\}$). The *residue field* of ν is the quotient $\kappa_\nu := \mathbf{R}_\nu / \mathbf{M}_\nu$. If $\mathbf{K}|\mathbb{F}$ is an extension of fields such that $\nu|\mathbb{F} = 0$ (or equivalently, if $\mathbf{R}_\nu$ is a $\mathbb{F}$-algebra), we say that ν is a *valuation of* $\mathbf{K}|\mathbb{F}$; in this case, the residue field κ_ν is also an extension of $\mathbb{F}$ and the (Krull) *dimension* of ν over $\mathbb{F}$ is the transcendence degree of the extension $\kappa_\nu|\mathbb{F}$.

Recall that a subgroup Δ of Γ is called *isolated* if for every $\alpha \in \Delta$, the elements in the interval $[-\alpha, \alpha] \cap \Gamma$ belong to Δ. The *rank* of ν is the number $\mathrm{rank}(\nu)$ of isolated subgroups of Γ. The *rational rank* of ν is the rational rank of Γ; i.e., is the number $\mathrm{rat.rank}(\nu)$ defined to be the dimension over $\mathbb{Q}$ of the vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. We have that $\mathrm{rank}(\nu) \leq \mathrm{rat.rank}(\nu)$, and for an extension of fields $\mathbf{K}|\mathbb{F}$,

$$\mathrm{rat.rank}(\nu) + \dim(\nu) \leq \mathrm{tr.deg}(\mathbf{K}|\mathbb{F})\,. \tag{2.1}$$

Now suppose that $\mathbf{K}$ is a function field over a field $\mathbb{F}$ of transcendence degree d. A *prime divisor* of $\mathbf{K}$ over $\mathbb{F}$ is by definition a dimension $d-1$ valuation ν of $\mathbf{K}|\mathbb{F}$. In this case (2.1) implies $\mathrm{rank}(\nu) = 1$, and so ν is in fact a discrete valuation; i.e., its value group is isomorphic to $\mathbb{Z}$ and its residue field is a finitely generated extension of $\mathbb{F}$.

Finally, let $\mathcal{X}$ be an irreducible algebraic variety over $\mathbb{F}$, and let $\mathbf{K}$ be its function field over $\mathbb{F}$. A valuation ν of $\mathbf{K}|\mathbb{F}$ is said to be *centered* at a point $P \in \mathcal{X}$ if the local ring $\mathcal{O}_P(\mathcal{X})$ of $\mathcal{X}$ at P is dominated by the valuation ring of ν; i.e., whenever $\mathcal{O}_P(\mathcal{X}) \subseteq \mathbf{R}_\nu$ and $\mathbf{M}_\nu \cap \mathcal{O}_P(\mathcal{X}) = \mathcal{M}_P(\mathcal{X})$, the maximal ideal of $\mathcal{O}_P(\mathcal{X})$.

3. A generalization of a near weight function

Throughout, let $\Lambda = (\Lambda, +, \preceq)$ be an (additive, abelian, cancelative) semigroup equipped with an admissible and total order $\preceq$. We write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$. Let $\Lambda_{-\infty} :=$

$\Lambda \cup \{-\infty\}$ with $-\infty \notin \Lambda$ being a symbol. We extend $\preceq$ to $\Lambda_{-\infty}$ by means of the rules $-\infty \preceq \alpha$, and $(-\infty) + \alpha = \alpha + (-\infty) = -\infty$ for any $\alpha \in \Lambda_{-\infty}$. Let $\mathbb{F}$ be a field, set $\mathbb{F}^* := \mathbb{F} \backslash \{0\}$, and let $\mathbf{R}$ be a (commutative) $\mathbb{F}$-algebra with 1. For a function $\rho : \mathbf{R} \to \Lambda_{-\infty}$ we consider the following sets

$$\mathcal{U}_\rho := \{f \in \mathbf{R} : \rho(f) \preceq \rho(1)\}\,, \quad \mathcal{M}_\rho := \{f \in \mathbf{R} : \rho(1) \prec \rho(f)\}\,.$$

Definition 3.1. Let Λ, $\mathbb{F}$, and $\mathbf{R}$ be as above. A function $\rho : \mathbf{R} \to \Lambda_{-\infty}$ is a *near order function* on $\mathbf{R}$ with value semigroup Λ if the following conditions hold true:

- (N0) $\rho(f) = -\infty$ if and only if $f = 0$;
- (N1) $\rho(\alpha f) = \rho(f)$ for all $\alpha \in \mathbb{F}^*$;
- (N2) $\rho(f + g) \preceq \max\{\rho(f), \rho(g)\}$;
- (N3) $\rho(f) \prec \rho(g)$ implies $\rho(fh) \preceq \rho(gh)$ but $\rho(fh) \prec \rho(gh)$ for $h \in \mathcal{M}_\rho$;
- (N4) $\rho(f) = \rho(g)$ and $f, g \in \mathcal{M}_\rho$, implies $\rho(f - \alpha g) \prec \rho(g)$ for some $\alpha \in \mathbb{F}^*$.

In this case $(\mathbf{R}, \rho, \Lambda)$ is said to be a *near order structure* on $\mathbf{R}$. We say that ρ is a *near weight function* on $\mathbf{R}$ if in addition ρ satisfies:

- (N5) $\rho(fg) \preceq \rho(f) + \rho(g)$, where equality holds for $f, g \in \mathcal{M}_\rho$.

In this case, we say that $(\mathbf{R}, \rho, \Lambda)$ defines a *near weight structure* on $\mathbf{R}$.

One can generalize the proof in [3, Lemmas 3,4] and state some direct consequences of Definition 3.1, namely:

Lemma 3.2. *Let $(\mathbf{R}, \rho, \Lambda)$ be a near order structure on a $\mathbb{F}$-algebra $\mathbf{R}$. Then:*

1. *$\mathcal{M}_\rho$ has no zero divisors*;
2. *If $\rho(f) \neq \rho(g)$, then $\rho(f + g) = \max\{\rho(f), \rho(g)\}$*;
3. *If $f, g, h \in \mathcal{M}_\rho$ and $\rho(f) = \rho(g)$, then $\rho(fh) = \rho(gh)$*;
4. *The element $\alpha \in \mathbb{F}^*$ in axiom (N4) above is unique.*

Remark 3.3. We point out that for a near weight function $\rho : \mathbf{R} \to \Lambda$, the set $\mathcal{U}_\rho$ is indeed a subalgebra of $\mathbf{R}$ by Axiom (N5).

Remark 3.4. If a near order structure $(\mathbf{R}, \rho, \Lambda)$ is such that ρ is surjective with $\mathcal{U}_\rho = \mathbb{F}$, then ρ was called an order function in [8, Def. 2.1].

3.1. Near weight functions from valuations. Here we construct near weight functions via valuations. The following result is our starting point.

Example 3.5. ([3, Ex. 3]) Let $\mathcal{X}$ be a (projective, nonsingular, geometrically irreducible, algebraic) curve defined over a finite field $\mathbb{F}$. Let $P_1, \ldots, P_m$ be pairwise different $\mathbb{F}$-rational points of $\mathcal{X}$, and let $\mathbf{R}$ be the $\mathbb{F}$-algebra of regular functions on $\Delta = \mathcal{X} \setminus \{P_1, \ldots, P_m\}$; i.e., $\mathbf{R} = \cap_{P \in \Delta} \mathcal{O}_P(\mathcal{X})$, where $\mathcal{O}_P(\mathcal{X})$ is the local ring of $\mathcal{X}$ at P. For each i, let ν_i be the valuation at P_i. Then the function $\rho_i : \mathbf{R} \to \mathbb{N}_0 \cup \{-\infty\}$ defined by

$$f \mapsto \begin{cases} -\infty & \text{if } f = 0\,, \\ 0 & \text{if } 0 \leq \nu_i(f)\,, \\ -\nu_i(f) & \text{if } \nu_i(f) < 0\,, \end{cases}$$

defines a near weight function on $\mathbf{R}$ with value semigroup $\mathbb{N}_0$.

We generalize this example as follows .

Example 3.6. (cf. [12]) Let $\mathcal{X}$ be a projective, geometrically irreducible, algebraic variety defined over a field $\mathbb{F}$. Let d be its (Krull) dimension. Let $C_1, \ldots, C_k$ be pairwise different codimension one subvarieties of $\mathcal{X}$, and let $\nu_i : \mathbb{F}(\mathcal{X}) \to \Gamma_i \cup \{\infty\}$ be valuations of the function field $\mathbb{F}(\mathcal{X})$ of $\mathcal{X}$ such that

(1) the rational rank of each ν_i equals d;
(2) ν_i is centered at a non-singular point $P_i \in \mathcal{X}(\mathbb{F}) \cap C_i$.

Let $\mathbf{R} := \cap_{\{C \neq C_1, \ldots, C_k\}} \mathcal{O}_C$, where C range over all codimension one subvarieties of $\mathcal{X}$ different from $C_1, \ldots, C_k$, and $\mathcal{O}_C$ denotes the local ring of $\mathcal{X}$ at C. Let

$$\Lambda_i := \{-\nu_i(f) : f \in \mathbf{R}, \nu_i(f) \prec_i 0\} \cup \{0\}\,.$$

Then the function $\rho_i : \mathbf{R} \to \Lambda_i \cup \{-\infty\}$ defined by

$$f \mapsto \begin{cases} -\infty & \text{if } f = 0\,, \\ 0 & \text{if } 0 \preceq \nu_i(f)\,, \\ -\nu_i(f) & \text{if } \nu_i(f) \prec 0\,, \end{cases}$$

is a near weight order function on $\mathbf{R}$ with value semigroup Λ_i. Indeed, by the very definition of ν_i, each Λ_i is in fact an ordered semigroup. From (2.1), $\dim(\nu_i) = 0$. Then, by condition (2), the residue field κ_i of ν_i coincides with $\mathbb{F}$. Thus Axioms (N0), (N1), (N2), (N3) and (N5) in Definition 3.1 clearly follow since ν_i is a valuation. Now we prove Axiom (N4). Let $f, g \in \mathcal{M}_{\rho_i}$ be such that $\rho_i(f) = \rho_i(g)$. Then $-\nu_i(f) = -\nu_i(g)$, and then we get $\nu_i(f/g) = 0$. Since $\kappa_i \cong \mathbb{F}$, there exists $\lambda_i \in \mathbb{F}$ such that $\nu_i(f/g - \lambda_i) \succ_i 0$. Thus $\rho_i(f - \lambda_i g) \prec_i \rho_i(g)$ and ρ_i is in fact a near weight function on $\mathbf{R}$.

Example 3.7. (cf. [12]) Let $\mathcal{X}$ and d be as in Example 3.6. Let $C_1, \ldots, C_k$ be pairwise different irreducible codimension one subvarieties of $\mathcal{X}$. For each $i = 1, \ldots, k$ we consider the following flag of subvarieties of $\mathcal{X}$:

$$\mathcal{F}_i : \quad \mathcal{X} = V_{i_0} \supseteq V_{i_1} := C_i \supseteq V_{i_2} \supseteq \ldots \supseteq V_{i_{d-1}} \supseteq V_{i_d} ,$$

where each V_{i_j} is an irreducible codimension one subvariety of $V_{i_{j-1}}$.

Then each rational function g on $V_{i_{j-1}}$ has a well-defined order along V_{i_j} which we denote by $\nu_{i_j}(g)$. Therefore, we can associate to $\mathcal{F}_i$ the valuation $\nu_{\mathcal{F}_i}$ defined as follows. For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d\}$ fix a function g_{i_j} on $V_{i_{j-1}}$ with a zero of order 1 along V_{i_j}. Then for any $f \in \mathbb{F}(\mathcal{X})$, $f = g_{i_1}^{n_{i_1}} u_{i_1}$ with $n_{i_1} = \nu_{i_1}(f) \in \mathbb{Z}$, $u_{i_1} \in \mathbb{F}(V_{i_0})$ and $\nu_{i_1}(u_{i_1}) = 0$. So $\bar{u}_{i_1} \in \mathbb{F}(V_{i_1})$, and $\bar{u}_{i_1} = g_{i_2}^{n_{i_2}} u_{i_2}$ with $n_{i_2} = \nu_{i_2}(\bar{u}_{i_1}) \in \mathbb{Z}$, $u_{i_2} \in \mathbb{F}(V_{i_1})$ and $\nu_{i_2}(u_{i_2}) = 0$; then $\bar{u}_{i_2} \in \mathbb{F}(V_{i_2})$. Continuing this process, we have $\bar{u}_{i_{j-1}} = g_{i_j}^{n_{i_j}} u_{i_j}$ with $n_{i_j} = \nu_{i_j}(\bar{u}_{i_{j-1}}) \in \mathbb{Z}$, $u_{i_j} \in \mathbb{F}(V_{i_{j-1}})$ and $\nu_{i_j}(u_{i_j}) = 0$. Let

$$\nu_{\mathcal{F}_i}(f) = (n_{i_1}, \ldots, n_{i_d}) \in \mathbb{Z}^d .$$

Then $\nu_i := \nu_{\mathcal{F}_i}$ is a discrete valuation of $\mathbb{F}(\mathcal{X})$ whose rational rank and value group is d and $\mathbb{Z}^d$ (equipped e.g. with the lexicographic order lex), respectively.

Let $\mathbf{R} = \cap_{V \neq V_{i_1}} \mathcal{O}_V(\mathcal{X})$ be the subring of $\mathbb{F}(\mathcal{X})$ consisting of regular functions with poles along V_{i_1}, $i = 1, \ldots, k$. Then, for each $i = 1, \ldots, k$,

$$\Lambda_i := \{-\nu_i(f) : f \in \mathbf{R} \text{ and } \nu_i(f) \preceq_{lex} (0, \ldots, 0) =: \mathbf{0}\} \cup \{\mathbf{0}\}$$

is an ordered subsemigroup of $\mathbb{Z}^d$, and the function $\rho_i : \mathbf{R} \to \Lambda_i \cup \{-\infty\}$ defined by

$$\rho_i(f) = \begin{cases} -\infty & \text{if } f = 0 \,, \\ \mathbf{0} & \text{if } \mathbf{0} \preceq \nu_i(f) \,, \\ -\nu_i(f) & \text{if } \nu_i(f) \prec \mathbf{0} \,, \end{cases}$$

is a near weight functions on $\mathbf{R}$.

3.2. Near weight functions from toric rings. Here we construct near weight functions on certain toric rings. We begin by recalling some notation and concepts from Toric Ring Theory. Our main reference is [18, Ch. 4].

Let d, n be positive integers and $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^d$. Let $\mathbb{F}$ be a field. Let $\mathbf{M}_n$ be the set of monomials in the polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$ and consider the function

$$\omega : \mathbf{M}_n \to \mathbb{Z}^d \,, \quad \mathbf{X}^\alpha \mapsto \sum_{i=1}^{n} \alpha_i a_i \,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^d$ and $\mathbf{X}^\alpha := X_1^{\alpha_1} \ldots X_n^{\alpha_n}$. Then the image of ω is a semigroup in $\mathbb{Z}^d$, namely

$$\langle \mathcal{A} \rangle := \{\alpha_1 a_1 + \ldots + \alpha_n a_n : \alpha_i \in \mathbb{N}_0\}\,.$$

The binomial ideal

$$I_{\mathcal{A}} := (\mathbf{X}^\alpha - \mathbf{X}^\beta : \alpha, \beta \in \mathbb{N}_0^n,\ \omega(\mathbf{X}^\alpha) = \omega(\mathbf{X}^\beta))$$

of $\mathbb{F}[X_1, \ldots, X_n]$ is called the *toric ideal* related to $\mathcal{A}$; the quotient ring $\mathbb{F}[X_1, \ldots, X_n]/I_{\mathcal{A}}$ is called a *toric ring*.

Let $\preceq$ be a total order in $\mathbb{Z}^d$. A nonzero element $F \in \mathbb{F}[X_1, \ldots, X_n]$ can be written as $F = \sum_{\text{finite}} \lambda_\alpha \mathbf{X}^\alpha$, $\lambda_\alpha \in \mathbb{F}$ and $\alpha \in \mathbb{N}_0^n$. We extend the function ω above to $\mathbb{F}^*[X_1, \ldots, X_n]$ via the formula

$$F \mapsto \omega(F) = \max\{\omega(\mathbf{X}^\alpha) : \mathbf{X}^\alpha \in \mathrm{Supp}(F)\}\,,$$

where $\mathrm{Supp}(F) := \{\mathbf{X}^\alpha : \lambda_\alpha \neq 0\}$. Next we consider the lexicographic order ($<$) on $\mathbb{F}[X_1, \ldots, X_n]$ such that $X_n < \ldots < X_1$. Then we can find a reduced Gröbner bases $\mathcal{G}$ associated to $I_{\mathcal{A}}$, where the elements of $\mathcal{G}$ are of the form $\mathbf{X}^\alpha - \mathbf{X}^\beta$ with $\alpha, \beta \in \mathbb{N}_0^n$ and $\omega(\mathbf{X}^\alpha) = \omega(\mathbf{X}^\beta)$; see [18, Cor. 4.4].

Consider the toric ring $\mathbf{R} = \mathbb{F}[X_1, \ldots, X_n]/I_{\mathcal{A}}$, where $I_{\mathcal{A}} = (\mathcal{G})$, and let $\mathbf{x}^\alpha$ be the equivalent class of $\mathbf{X}^\alpha$ modulo $I_{\mathcal{A}}$. We define $\omega(\mathbf{x}^\alpha) := \omega(\mathbf{X}^\alpha)$ which is well defined as $\mathbf{x}^\alpha = \mathbf{x}^\beta$ if and only if $\mathbf{X}^\alpha - \mathbf{X}^\beta \in I_{\mathcal{A}}$ if and only if $\omega(\mathbf{X}^\alpha) = \omega(\mathbf{X}^\beta)$ by [18, Lemma 4.1]).

Let $\Delta(I_{\mathcal{A}})$ be the footprint of $I_{\mathcal{A}}$; i. e., $\Delta(I_{\mathcal{A}}) = \{\alpha \in \mathbb{N}_0^n : \mathbf{X}^\alpha \notin \mathrm{LT}(I_{\mathcal{A}})\}$, where $\mathrm{LT}(I_{\mathcal{A}})$ is the set of leading terms of elements of $I_{\mathcal{A}}$ with respect to the lexicographic order. Let $B = \{\mathbf{x}^\alpha : \alpha \in \Delta(I_{\mathcal{A}})\} \subseteq \mathbf{R}$. Then B is a basis for the $\mathbb{F}$-vector space $\mathbf{R}$, and each element $f \in \mathbf{R}$ can be written uniquely as $f = \sum_{\text{finite}} \lambda_\alpha \mathbf{x}^\alpha$ with $\lambda_\alpha \in \mathbb{F}$ and $\alpha \in \Delta(I_{\mathcal{A}})$ (see [5, Chap. 5, §3]). Then define

$$\omega(f) = \max_{\preceq}\{\omega(\mathbf{x}^\alpha) : \mathbf{x}^\alpha \in \mathrm{Supp}(f)\}\,.$$

Let $\Lambda := \{\gamma \in \mathbb{N}_0\mathcal{A} : \mathbf{0} := (0, \ldots, 0) \preceq \gamma\} \subseteq \mathbb{N}_0\mathcal{A}$ and suppose that $\preceq$ is an admissible total order on Λ; then $(\Lambda, \preceq)$ is an ordered semigroup.

Example 3.8. Notation as above. The function $\rho : \mathbf{R} \to \Lambda \cup \{-\infty\}$ defined by

$$f \mapsto \begin{cases} -\infty & \text{if } f = 0\,, \\ \mathbf{0} & \text{if } f \neq 0 \text{ and } \omega(f) \preceq \mathbf{0}\,, \\ \omega(f) & \text{if } \mathbf{0} \prec \omega(f)\,, \end{cases}$$

is a near weight function on $\mathbf{R}$. To see this, let $f, g \in \mathbf{R}$. Then $f = \sum_{\text{finite}} \lambda_\alpha \mathbf{x}^\alpha + \lambda_\beta \mathbf{x}^\beta$ and $g = \sum_{\text{finite}} \mu_\delta \mathbf{x}^\delta + \mu_\gamma \mathbf{x}^\gamma$, where $\lambda_\alpha, \lambda_\beta, \mu_\delta, \mu_\gamma \in \mathbb{F}$, and $\alpha, \beta, \delta, \gamma \in \Delta(I_{\mathcal{A}})$ with $\omega(\mathbf{x}^\alpha) \preceq$

$\omega(\mathbf{x}^\beta)$ and $\omega(\mathbf{x}^\delta) \preceq \omega(\mathbf{x}^\gamma)$ for all $\alpha, \delta \in \Delta(I_{\mathcal{A}})$. Note that if λ_β and μ_γ are nonzero and $\omega(\mathbf{x}^\beta) \succ \mathbf{0}$ and $\omega(\mathbf{x}^\gamma) \succ \mathbf{0}$, then $\rho(f) = \omega(\mathbf{x}^\beta)$ and $\rho(g) = \omega(\mathbf{x}^\gamma)$.

By the definition of ω, Axioms (N0), (N1) and (N2) in Definition 3.1 follow. We now prove (N4) and (N5) as Axiom (N3) is a consequence of (N5).

Suppose $\rho(f) = \rho(g)$ with $f, g \in \mathcal{M}_\rho$, then $\omega(f) = \omega(g)$, or else, $\omega(\mathbf{x}^\beta) = \omega(\mathbf{x}^\gamma)$. So $\mathbf{x}^\beta = \mathbf{x}^\gamma$ and taking $\lambda = \lambda_\beta/\mu_\gamma \in \mathbb{F}$ we have $\omega(f - \lambda g) = \max_{\preceq}\{\omega(\mathbf{x}^\alpha) : \mathbf{x}^\alpha \in \operatorname{supp}(f - \lambda g)\} \prec \omega(f)$. Therefore $\rho(f - \lambda g) \prec \rho(f)$ and Axiom (N4) follows. Finally, Axiom (N5) is a consequence of the fact that $\mathbf{x}^\beta \mathbf{x}^\gamma = \mathbf{x}^{\beta+\gamma}$ and that $\omega(\mathbf{x}^\beta \mathbf{x}^\gamma) = \omega(\mathbf{x}^\beta) + \omega(\mathbf{x}^\gamma)$.

Remark 3.9. If in the above notation $\mathcal{A} \subseteq \mathbb{N}_0^d$, then $\Lambda = \mathbb{N}_0\mathcal{A}$ is a well-ordered semigroup and ρ (cf. Example 3.8) is a weight function (see [16, §1]).

Example 3.10. Let us consider the particular case $\mathcal{A} = \{(-1,1),(0,1),(1,1)\} \subseteq \mathbb{Z}^2$. Let $\mathbf{M} = \mathbf{M}_3$ be the set of monomials in $\mathbb{F}[X,Y,Z]$ and consider the monomial function $\omega : \mathbf{M} \to \mathbb{Z}^2$ defined by $\omega(X) = (-1,1)$, $\omega(Y) = (1,1)$ and $\omega(Z) = (0,1)$. Then the toric ideal associated to $\mathcal{A}$ in $\mathbb{F}[X,Y,Z]$ is given by $I_{\mathcal{A}} = (XY - Z^2)$. Let $\mathbb{N}_0\mathcal{A} = \langle\{(-1,1),(0,1),(1,1)\}\rangle$ be the semigroup generated by $\mathcal{A}$ and let $\preceq$ be the lexicographic order on $\mathbb{Z}^2$. Let $\Lambda := \{\gamma \in \mathbb{N}_0\mathcal{A} : (0,0) \preceq \gamma\} = \langle\{(0,1),(1,1)\}\rangle$. Taking $\mathbf{R} = \mathbb{F}[X,Y,Z]/I_{\mathcal{A}}$ we have from Example 3.8 a near weight function $\rho_1 : \mathbf{R} \to \Lambda \cup \{-\infty\}$ with $\rho_1(x) = (0,0)$, $\rho_1(y) = (1,1)$ and $\rho_1(z) = (0,1)$ where x, y, z are the classes of X, Y, Z modulo $I_{\mathcal{A}}$. In this case,

$$\mathcal{U}_{\rho_1} = \{\sum_{finite} \lambda_{ab} x^a z^b : \lambda_{ab} \in \mathbb{F},\ a \neq 0 \text{ or } a = b = 0\}.$$

On the other hand, if we define ω by $\omega(X) = (1,1)$, $\omega(Y) = (-1,1)$ and $\omega(Z) = (0,1)$ we get the same toric ideal $I_{\mathcal{A}} = (XY - Z^2)$ in $\mathbb{F}[X,Y,Z]$ and then from Example 3.8 we can construct another near weight function $\rho_2 : \mathbf{R} \to \Lambda \cup \{-\infty\}$ with $\rho_2(x) = (1,1)$, $\rho_2(y) = (0,0)$ and $\rho_2(z) = (0,1)$. Here we have,

$$\mathcal{U}_{\rho_2} = \{\sum_{finite} \lambda_{bc} y^c z^b : \lambda_{bc} \in \mathbb{F},\ c \neq 0 \text{ or } c = b = 0\}.$$

4. Codes from an admissible set of near weights structures

In this section we construct linear codes from $\mathbb{F}$-algebras which admit a (finite) set of admissible near weight structures. Notation as in Section 3.

Definition 4.1. Let $(\mathbf{R}, \rho, \Lambda)$ be a near order structure. The normalization $\tilde{\rho}$ of ρ is the function $\tilde{\rho} : \mathbf{R} \to \Lambda_{-\infty}$ defined by $\tilde{\rho}(0) = -\infty$, $\tilde{\rho}(f) := 0$ if $f \in \mathcal{U}_\rho \setminus \{0\}$ and $\tilde{\rho}(f) := \rho(f)$ if $f \in \mathcal{M}_\rho$.

Then $\tilde{\rho}$ is also a near order function, where $\mathcal{U}_{\tilde{\rho}} = \mathcal{U}_\rho$ and $\mathcal{M}_{\tilde{\rho}} = \mathcal{M}_\rho$. From now on we assume that our near order functions are already normalized.

Definition 4.2. Let $\mathbf{R}$ be an $\mathbb{F}$-algebra. A set of pairwise different near weight structures $(\mathbf{R}, \rho_i, \Lambda_i)$, $i = 1, \ldots, m$, is said to be admissible whenever $\cap_{i=1}^m \mathcal{U}_{\rho_i} = \mathbb{F}$.

Example 4.3. The set of near weights ρ_1, ρ_2 in Example 3.10 is admissible.

Example 4.4. In Example 3.7, each $\mathcal{U}_{\rho_i}$ is contained in $\mathcal{O}_{V_{i_1}}(\mathcal{X}) \cap \mathbf{R}$ for $i \in \{1, \ldots, k\}$, where $\mathcal{O}_{V_{i_1}}(\mathcal{X})$ is the local ring associated to V_{i_1}. Since $\cap_{i=1}^k \mathcal{U}_{\rho_i} \subseteq \left(\cap_{i=1}^k \mathcal{O}_{V_{i_1}}(\mathcal{X})\right) \cap \mathbf{R} = \mathbb{F}$ (see [21, Ch.VI,§14]), we have that $(\mathbf{R}, \rho_i, \Lambda_i), i = 1, \ldots, k$ is an admissible set of near weight structures on $\mathcal{R}$.

From Lemma 3.2 we obtain the following.

Proposition 4.5. *Let* $\mathbf{R}$ *be an* $\mathbb{F}$*-algebra admitting a finite set of admissible near weight structures. Then* $\mathbf{R}$ *is an integral domain.*

For the remainder of this section we let $\mathbf{R}$ be an $\mathbb{F}$-algebra with $\mathbb{F}$ being a finite field. Let $(\mathbf{R}, \rho_1, \Lambda_1), \ldots, (\mathbf{R}, \rho_m, \Lambda_m)$ be a set of pairwise different admissible near weight structures on $\mathbf{R}$. Let $\bar{\Lambda} := \prod_{i=1}^m \Lambda_i$. We have a natural partial order on $\bar{\Lambda}$; indeed, for $\alpha = (\alpha_\mathbf{1}, \ldots, \alpha_\mathbf{m}) \in \bar{\mathbf{\Lambda}}$, $\beta = (\beta_\mathbf{1}, \ldots, \beta_\mathbf{m}) \in \bar{\mathbf{\Lambda}}$, $\alpha \preceq \beta$ if $\alpha_i \preceq_i \beta_i$ for each i. We define

$$\mathcal{L}(\alpha) = \{f \in \mathbf{R} : \rho_i(f) \preceq_i \alpha_i,\, i = 1, \ldots, m\}\,.$$

From axioms (N0), (N1), (N2) in Definition 3.1, $\mathcal{L}(\alpha)$ is in fact a $\mathbb{F}$-vector subspace of $\mathbf{R}$. We also observe that $\mathcal{L}(0) = \mathbb{F}$, and $\mathcal{L}(\alpha) \subseteq \mathcal{L}(\beta)$ provided that $\alpha \preceq \beta$.

Let $\varphi : \mathbf{R} \to \mathbb{F}^n$ be a epimorphism of $\mathbb{F}$-algebras. For each $\alpha \in \bar{\Lambda}$ we define the code

$$E(\alpha) := \varphi(\mathcal{L}(\alpha))\,,$$

and we shall determine a lower bound on the minimum distance of the dual code $C(\alpha) := (E(\alpha))^\perp$; we proceed as in [4, §2]. Consider the following subset of $\bar{\Lambda}$:

$$\mathcal{H} = \mathcal{H}(\rho_1, \ldots, \rho_m) = \{(\rho_1(f), \ldots, \rho_m(f)) : f \in \mathbf{R}\backslash\{0\}\}\,.$$

Let $\alpha_1 = (0, \ldots, 0) \in \mathcal{H}$ and choose a strictly increasing sequence $\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots$ of elements of $\mathcal{H}$ (with respect to the partial order $\preceq$ on $\bar{\Lambda}$) such that

$$\dim_\mathbb{F}(E(\alpha_{j+1})/E(\alpha_j)) = 1\,.$$

Definition 4.6. Let $N(\alpha_j) := \{(f_{j,k}, g_{j,k}) : k = 1, \ldots, \ell_j\}$ be a set of pairs of functions in $\mathbf{R} \times \mathbf{R}$ such that:

(a) $f_{j,k}, g_{j,k} \in \mathcal{L}(\alpha_{j+1})$;

(b) $f_{j,k}g_{j,k} \in \mathcal{L}(\alpha_{j+1})\backslash\mathcal{L}(\alpha_j)$ (hence $\rho_i(f_{j,k}) + \rho_i(g_{j,k}) = \alpha_{(j+1)i}$, for some $i \in \{1,\dots,m\}$);
(c) for i as in (b), $\rho_i(f_{j,1}) \prec_i \dots \prec_i \rho_i(f_{j,\ell_j})$ (and therefore $\rho_i(g_{j,1}) \succ_i \dots \succ_i \rho_i(g_{j,\ell_j})$);
(d) given $s \in \{1,\dots,\ell_j - 1\}$, $f_{j,s}g_{j,r} \in \mathcal{L}(\alpha_j)$ for $r = s+1,\dots,\ell_j$.

We set $\mu(\alpha_j) := \#N(\alpha_j)$.

Let M and N be the matrices where the first ℓ_j rows of M are $\varphi(f_{j,1}),\dots,\varphi(f_{j,\ell_j})$, and the first ℓ_j columns of N are $\varphi(g_{j,1}),\dots,\varphi(g_{j,\ell_j})$; we complete the rows of M and the columns of N in a such way that the rank of M and N are equal to n. For $y = (y_1,\dots,y_n) \in \mathbb{F}^n$ consider the diagonal matrix $D(y) := (a_{sr})_{n\times n}$ where $a_{sr} = 0$ if $r \neq s$ and $a_{rr} = y_r$ for $r,s = 1,\dots,n$. Let $S(y) := MD(y)N$. Then for $r,s \in \{1,\dots,\ell_j\}$ we get $(S(y))_{s,r} = y \cdot (\varphi(f_{j,s}) * \varphi(g_{j,r}))$ where $\cdot$ is the usual inner product in $\mathbb{F}^n$ and $*$ is the usual component wise product that makes $\mathbb{F}^n$ an $\mathbb{F}$-algebra. Since $\mathrm{rank}(M) = \mathrm{rank}(N) = n$, $\mathrm{rank}(S(y)) = \mathrm{wt}(y)$ where $\mathrm{wt}(y)$ is the weight of y.

Proposition 4.7. *If* $y \in C(\alpha_j) \setminus C(\alpha_{j+1})$*, then* $\mathrm{wt}(y) \geq \mu(\alpha_j)$.

Proof. Note that $(S(y))_{s,r} = y \cdot (\varphi(f_{j,s}) * \varphi(g_{j,r})) = y \cdot \varphi(f_{j,s}g_{j,r})$ for all $r,s \in \{1,\dots,\ell_j\}$. From Definition 4.6 we have that $f_{j,s}g_{j,r} \in \mathcal{L}(\alpha_j)$ if $s < r$ and $f_{j,r}g_{j,r} \in \mathcal{L}(\alpha_{j+1})\backslash\mathcal{L}(\alpha_j)$. Then for $y \in C(\alpha_j) \setminus C(\alpha_{j+1})$ we get $(S(y))_{s,r} = 0$ if $s < r$ and from hypothesis $\dim_{\mathbb{F}}(E(\alpha_{j+1})/E(\alpha_j)) = 1$ we get $(S(y))_{r,r} = y \cdot \varphi(f_{j,r}g_{j,r}) \neq 0$. Therefore $\mathrm{rank}(S(y)) \geq \ell_j = \mu(\alpha_j)$. ☐

Since $E(\alpha_j) \subsetneq E(\alpha_{j+1}) \subseteq \mathbb{F}^n$, there exists a positive integer $\bar{N}$ such that $E(\alpha_{\bar{N}}) = \mathbb{F}^n$. As a consequence of Proposition 4.7 we have the following bound on the minimum distance of $C(\alpha)$.

Corollary 4.8. *For* $j = 1,\dots,\bar{N}$, $d(C(\alpha_j)) \geq d(\alpha_j) := \min\{\mu(\alpha_k) : j \leq k, k = 1,\dots,\bar{N}\}$.

Example 4.9. Notation as above. Let $\mathbf{R} = \mathbb{F}_3[X,Y,Z]/(XY - Z^2)$ be the toric ring worked out in Example 3.10 over the finite field $\mathbb{F}_3$. From example 4.4 we have that $\{(\mathbf{R},\rho_1,\Lambda),(\mathbf{R},\rho_2,\Lambda)\}$ is an admissible set of near weights structures on $\mathbf{R}$ where $\Lambda = \langle\{(0,1),(1,1)\}\rangle$,

$$\begin{array}{c}\rho_1(x) = (0,0),\ \rho_1(y) = (1,1),\ \rho_1(z) = (0,1), \text{ and}\\ \rho_2(x) = (1,1),\ \rho_2(y) = (0,0),\ \rho_2(z) = (0,1)\ .\end{array}$$

The set of rational points of the toric variety $\mathcal{V}_{\mathbb{F}_3}(XY - Z^2)$ is $\{P_1 = (0,0,0), P_2 = (0,1,0), P_3 = (1,0,0), P_4 = (1,1,1), P_5 = (0,2,0), P_6 = (2,0,0), P_7 = (1,1,2), P_8 =$

$(2,2,1), P_9 = (2,2,2)\}$. Then taking the epimorphism $\varphi : \mathbf{R} \to \mathbb{F}_3^9$ defined by $\varphi(f) = (f(P_1), \ldots, f(P_9))$ we can construct the codes $E(\alpha) := \varphi(\mathcal{L}(\alpha))$ and $C(\alpha) := (E(\alpha))^{\perp}$ for $\alpha \in \mathcal{H}$. In Table 1 we list all the values $\mu(\alpha_j)$ and $d(\alpha_j)$ for $j = 1, \ldots, 9$.

j	$\alpha_j = (\alpha_{j1}, \alpha_{j2})$	$\mu(\alpha_j)$	$d(\alpha_j)$
1	(0,0,0,0)	2	2
2	(0,1,0,1)	3	2
3	(0,2,0,2)	2	2
4	(1,1,0,2)	2	2
5	(1,1,1,1)	4	3
6	(1,2,1,2)	6	3
7	(1,2,1,3)	3	3
8	(2,2,1,3)	3	3
9	(2,2,2,2)	6	4

TABLE 1. Bound $d(\alpha_j)$ of code $C(\alpha_j)$ using the order $\prec_{lex}$.

Now, if in Λ we consider the graded reverse lexicographic order $\prec_{lgr}$ we will still have that $(\mathbf{R}, \rho_1, \Lambda), (\mathbf{R}, \rho_2, \Lambda)$ is an admissible set of near weights structures on $\mathbf{R}$; in Table 2 we present results for the parameters $\mu(\alpha_j)$ and $d(\alpha_j)$ for $j = 1, \ldots, 9$.

j	$\alpha_j = (\alpha_{j1}, \alpha_{j2})$	$\mu(\alpha_j)$	$d(\alpha_j)$
1	(0,0,0,0)	2	2
2	(0,1,0,1)	2	2
3	(1,1,0,1)	2	2
4	(1,1,1,1)	3	3
5	(0,2,0,2)	4	3
6	(0,2,1,2)	3	3
7	(2,2,1,2)	3	3
8	(2,2,2,2)	4	4
9	(1,3,2,2)	4	4

TABLE 2. bound $d(\alpha_j)$ of code $C(\alpha_j)$ using the order $\prec_{lgr}$.

Remark 4.10. Goppa codes supported in m points can be obtained from the codes described above [4, Thm. 2.10]. Let $\mathcal{X}$ be a nonsingular, geometrically irreducible, projective algebraic curve defined over a finite field $\mathbb{F}$, and let $G := \sum_{i=1}^{m} a_i Q_i$ and $D := P_1 + \ldots + P_n$ be divisors on $\mathcal{X}$ such that $\mathrm{Supp}(G) \cap \mathrm{Supp}(D) = \emptyset$ and such that each P_i is a $\mathbb{F}$-rational point. Then the Goppa code $C_{\mathcal{L}}(D, G)$ is the set of m-tuples $(h(P_1), \ldots, h(P_n))$, where $h \in \mathcal{L}(G)$ (the Riemann-Roch space associated to G). Let $\mathbf{R} = \cap_{Q \in \mathcal{X} \setminus \{Q_1, \ldots, Q_m\}} \mathcal{O}_Q(\mathcal{X})$, where $\mathcal{O}_Q(\mathcal{X})$ is the local ring at $Q \in \mathcal{X}$. By defining $\varphi(f) := (f(P_1), \ldots, f(P_n))$, there exist an admissible set of m near weight structures on $\mathcal{R}$ (as in Example 3.5) such that $C_{\mathcal{L}}(D,G) = E(a)$ (and thus $C_{\mathcal{L}}(D,G)^{\perp} = C(a)$), where $a = (a_1, \ldots, a_m)$.

Example 4.11. Let $\mathcal{X}$ be the Hermitian curve $X^5 = ZY^4 + Z^4Y$ over $\mathbb{F}_{16}$. Let $Q_1, Q_2 \in \mathcal{X}$ be two distinct $\mathbb{F}_{16}$-rational points of $\mathcal{X}$, $a = (a_1, a_2) \in \mathbb{N}_0^2$ and denote by $C_{\mathcal{L}}(D,G)$ the Goppa code associated to the divisors $G := a_1Q_1 + a_2Q_2$ and $D := P_1 + \ldots + P_n$, where $P_1, \ldots, P_n$ are distinct $\mathbb{F}_{16}$-rational points, different from Q_1 and Q_2. Since the genus of $\mathcal{X}$ is $g = 6$, the Goppa bound for the minimum distance of the code $C_{\mathcal{L}}(D,G)^{\perp} = C(a)$ is $d_G(a) = a_1 + a_2 - (2g-2) = a_1 + a_2 - 10$. Table 3 compares the bounds $d(a)$ and $d_G(a)$ for the minimum distance of codes $C(a)$.

j	$a_j := a = (a_1, a_2)$	$\mu(a_j)$	$d(a_j)$	$d_G(a_j)$
1	(0,0)	2	2	-10
2	(3,3)	2	2	-4
3	(4,3)	2	2	-3
4	(4,4)	2	2	-2
5	(4,5)	2	2	-1
6	(5,5)	3	3	0
7	(6,6)	4	4	2
8	(7,6)	4	4	3
9	(7,7)	5	4	4

TABLE 3. Bounds $d(a)$ and $d_G(a)$ for the code $C(a)$.

5. Characterizing algebras that admit two well-agreeing near weight structures

In this section we present a characterization for algebras which admit two admissible near weight structures. As a matter of fact, under certain conditions, we will see that such algebras are simply the rings of regular functions of affine irreducible algebraic varieties with two irreducible divisors at infinity. A basic reference here is [14].

Throughout, let $(\mathbf{R}, \rho_i, \Lambda_i)$, $i = 1, 2$ be two admissible near weight structures on an $\mathbb{F}$-algebra $\mathbf{R}$. Let $\bar{\Lambda} := \Lambda_1 \oplus \Lambda_2$ be the direct sum of Λ_1 and Λ_2 which naturally became a semigroup with the obvious structure. Consider the following subset of $\bar{\Lambda}$,

$$\mathcal{H} = \mathcal{H}(\rho_1, \rho_2) = \{(\rho_1(f), \rho_2(f)) : f \in \mathbf{R} \backslash \{0\}\}. \tag{5.1}$$

For $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2) \in \bar{\Lambda}$, define the *least upper bound of* $\mathbf{a}$ and $\mathbf{b}$ as being

$$\mathrm{Lub}(\mathbf{a}, \mathbf{b}) := (\max_{\preceq_1}\{a_1, b_1\}, \max_{\preceq_2}\{a_2, b_2\}).$$

Lemma 5.1. *If* $\mathbf{a}, \mathbf{b} \in \mathcal{H}$, *then* $\mathrm{Lub}(\mathbf{a}, \mathbf{b}) \in \mathcal{H}$. *Furthermore, if* $f, g \in \mathbf{R}$ *are such that* $\mathbf{a} = (\rho_1(f), \rho_2(f))$ *and* $\mathbf{b} = (\rho_1(g), \rho_2(g))$, *then there exist* $\lambda, \mu \in \mathbb{F}$ *such that* $\mathrm{Lub}(a, b) = (\rho_1(\lambda f + \mu g), \rho_2(\lambda f + \mu g))$.

Proof. Similar to the proof of [4, Prop. 3.3], [14, Lemma 3.1]. □

Proposition 5.2. *The set* $\mathcal{H}$ *in (5.1) is a subsemigroup of* $\bar{\Lambda}$.

Proof. Let $f, g \in \mathbf{R}$ such that $\mathbf{a} = (\rho_1(f), \rho_2(f))$, $\mathbf{b} = (\rho_1(g), \rho_2(g)) \in \mathcal{H}$. Let $c = (\rho_1(fg), \rho_2(fg)) \in \mathcal{H}$. From Axiom (N5) in Definition 3.1, $\rho_i(fg) \preceq_i \rho_i(f) + \rho_i(g)$ and equality holds whenever $\rho_i(f) \succ_i 0$ and $\rho_i(g) \succ_i 0$. Therefore, $a + b = \mathrm{Lub}(\mathrm{Lub}(a, b), c) \in \mathcal{H}$. □

Now assume that Λ_1 and Λ_2 are well ordered semigroups. For $a_i \in \Lambda_i$, $i = 1, 2$, let

$$x_1(a_2) := \min_{\preceq_1}\{a \in \Lambda_1 : (a, a_2) \in \mathcal{H}\}, \quad x_2(a_1) := \min_{\preceq_2}\{a \in \Lambda_2 : (a_1, a) \in \mathcal{H}\}.$$

Then if $a \in \Lambda_i$ and $x_j(a) \succ_j 0$, $x_i(x_j(a)) = a \succ_i 0$, for $i, j = 1, 2, i \neq j$ (this is similar to [14, Lemma 3.4]).

Consider the followings subsemigroups of Λ_1 and Λ_2, respectively:

$$\mathcal{H}_{\rho_1} := \rho_1(\mathcal{U}_{\rho_2}) = \{\alpha \in \Lambda_1 : (\alpha, 0) \in \mathcal{H}\}, \quad \mathcal{H}_{\rho_2} := \rho_2(\mathcal{U}_{\rho_1}) = \{\alpha \in \Lambda_2 : (0, \alpha) \in \mathcal{H}\}.$$

Lemma 5.3. *It holds that* $\alpha \in \Lambda_i \backslash \mathcal{H}_{\rho_i}$ *if and only if* $x_j(\alpha) \in \Lambda_j \backslash \mathcal{H}_{\rho_j}$ *for* $i, j = 1, 2$, $i \neq j$.

Proof. Let $\alpha \in \Lambda_1 \backslash \mathcal{H}_{\rho_1}$. Then $\alpha \neq 0$ and $(\alpha, 0) \notin \mathcal{H}_{\rho_1}$. Thus $x_2(\alpha) \succ_2 0$ and therefore $x_2(\alpha) \in \Lambda_2 \backslash \mathcal{H}_{\rho_2}$. Now, if $\alpha \in \Lambda_1$ and $x_2(\alpha) \in \Lambda_2 \backslash \mathcal{H}_{\rho_2}$ we have $x_2(\alpha) \succ_2 0$, and then $\alpha = x_1(x_2(\alpha)) \succ_1 0$. Therefore $\alpha \in \Lambda_1 \backslash \mathcal{H}_{\rho_1}$. Analogously, $\alpha \in \Lambda_2 \backslash \mathcal{H}_{\rho_2}$ if and only if $x_1(\alpha) \in \Lambda_1 \backslash \mathcal{H}_{\rho_1}$. □

Let $\mathcal{H}$ be defined in (5.1) and consider the following subset of $\mathcal{H}$

$$\Omega := \{(\alpha_1, x_2(\alpha_1)) : \alpha_1 \in \Lambda_1 \backslash \mathcal{H}_{\rho_1}\} \cup \{(\alpha_1, 0) : \alpha_1 \in \mathcal{H}_{\rho_1}\} \cup \{(0, \alpha_2) : \alpha_2 \in \mathcal{H}_{\rho_2}\}.$$

Note that

$$\{(\alpha_1, x_2(\alpha_1)) : \alpha_1 \in \Lambda_1 \backslash \mathcal{H}_{\rho_1}\} = \{(x_1(\alpha_2), \alpha_2) : \alpha_2 \in \Lambda_2 \backslash \mathcal{H}_{\rho_2}\}.$$

Proposition 5.4. $\mathcal{H} = \{\mathrm{Lub}(a, b) : a, b \in \Omega\}$.

Proof. Similar to the proof of [14, Prop. 3.6]. □

For each $a \in \mathcal{H}$ let $f_a \in \mathbf{R} \backslash \{0\}$ be such that $a = (\rho_1(f_a), \rho_2(f_a))$ and let $f_{\mathbf{0}} = 1$.

Proposition 5.5. *The set* $\mathcal{B} = \{f_a \in \mathbf{R} \backslash \{0\} : a \in \Omega\}$ *is a* $\mathbb{F}$*-basis of* $\mathbf{R}$.

Proof. We first show that $\mathcal{B}$ is $\mathbb{F}$-linearly independent. Let $\lambda_a \in \mathbb{F}$ such that $\sum_{finite} \lambda_a f_a = 0$. Since $a \in \Omega$, then

$$0 = \sum_{finite} \lambda_a f_a = \sum_{finite} \lambda_{(\alpha_1,0)} f_{(\alpha_1,0)} + \sum_{finite} \lambda_{(0,\alpha_2)} f_{(0,\alpha_2)} + \sum_{finite} \lambda_{(\alpha,x_2(\alpha))} f_\alpha$$

so that

$$\rho_1(\sum_{finite} \lambda_{(\alpha_1,0)} f_{(\alpha_1,0)} + \sum_{finite} \lambda_{(\alpha,x_2(\alpha))} f_\alpha) = \rho_1(\sum_{finite} \lambda_{(0,\alpha_2)} f_{(0,\alpha_2)}) = 0 \ .$$

From Axioms (N1) and (N2) in Definition 3.1, we get $\lambda_{(\alpha_1,0)} = \lambda_{(\alpha,x_2(\alpha))} = 0$ for $\alpha_1 \succ_1 0$ and $\alpha \succ_1 0$. Since $f_{\mathbf{0}} = 1$, then

$$\lambda_{\mathbf{0}} + \sum_{finite} \lambda_{(0,\alpha_2)} f_{(0,\alpha_2)} = 0\,.$$

Now, applying ρ_2 to the equality above, we get $\lambda_{(0,\alpha_2)} = 0$ so that $\lambda_{\mathbf{0}} = 0$. Therefore $\mathcal{B}$ is a linearly independent set.

We show next that $\mathcal{B}$ span $\mathbf{R}$. Let $f \in \mathbf{R}\backslash\{0\}$. Suppose that $\rho_2(f) = 0$ and we use induction on $\rho_1(f) = \alpha_1$. If $\alpha_1 = 0$, then $f \in \cap_{i=1,2}\mathcal{U}_{\rho_i} = \mathbb{F}$. Then, suppose by induction that for all element $g \in \mathbf{R}$ such that $\rho_1(g) \prec_1 \alpha_1$ and $\rho_2(g) = 0$ we have that g is generated by $\mathcal{B}$. As $(\alpha_1, 0) \in \Omega$, take $f_{(\alpha_1,0)} \in \mathcal{B}$. So $\rho_1(f) = \alpha_1 = \rho_1(f_{(\alpha_1,0)})$ and from Axiom (N4) in Definition 3.1 there exists $\lambda_1 \in \mathbb{F}$ such that $\rho_1(f - \lambda_1 f_{(\alpha_1,0)}) \prec_1 \alpha_1$ and $\rho_2(f - \lambda_1 f_{(\alpha_1,0)}) = 0$. Then by induction $f - \lambda_1 f_{(\alpha_1,0)}$ is generated by $\mathcal{B}$, and so f is.

Now, we use induction on $\rho_2(f) = \alpha_2$. If $\alpha_2 = 0$ the result follows from the previous case. By induction, suppose that for all element $g \in \mathbf{R}$ such that $\rho_2(g) \prec_2 \alpha_2$ and $\rho_1(g) \preceq_1 \rho_1(f)$ we have that g is generated by $\mathcal{B}$. Take $f_{(x_1(\alpha_2),\alpha_2)} \in \mathcal{B}$. As $\rho_2(f) = \alpha_2 = \rho_2(f_{(x_1(\alpha_2),\alpha_2)})$, from the aforementioned Axiom (N4) there exists $\lambda_2 \in \mathbb{F}$ such that $\rho_2(f - \lambda_2 f_{(x_1(\alpha_2),\alpha_2)}) \prec_2 \alpha_2$ and $\rho_1(f - \lambda_2 f_{(x_1(\alpha_2),\alpha_2)}) \preceq_1 \rho_1(f)$. By induction, $f - \lambda_2 f_{(x_1(\alpha_2),\alpha_2)}$ is generated by $\mathcal{B}$, and therefore f is so. $\square$

From now on, we assume that each semigroup Λ_i is finitely generated, says

$$\Lambda_i = \langle \alpha_{i1}, \ldots, \alpha_{im_i} \rangle \ ,$$

where $m_i \in \mathbb{N}$. In particular, each Λ_i is a well ordered semigroup; see [16, §1]. Consider the following subset of $\mathcal{H}$,

$$\bar{\Omega} := \{(\alpha_{1t}, x_2(\alpha_{1t})) : 1 \leq t \leq m_1\} \cup \{(x_1(\alpha_{2k}), \alpha_{2k}) : 1 \leq k \leq m_2\}\,. \tag{5.2}$$

For $i = 1, 2$ and $j = 1, \dots, m_i$, let $f_{\alpha_{ij}} \in \mathbf{R}$ such that $\rho_i(f_{\alpha_{ij}}) = \alpha_{ij}$ and $\rho_l(f_{\alpha_{ij}}) = x_l(\alpha_{ij})$, $\ell \in \{1, 2\}$ and $\ell \neq i$. Then given $\gamma \in \Lambda_i$, there exists $\lambda_j \in \mathbb{N}_0$ such that $\gamma = \sum_{j=1}^{m_i} \lambda_j \alpha_{ij}$, and hence

$$\gamma = \sum_{j=1}^{m_i} \lambda_j \alpha_{ij} = \sum_{j=1}^{m_i} \lambda_j \rho_i(f_{\alpha_{ij}}) = \sum_{j=1}^{m_i} \rho_i(f_{\alpha_{ij}}^{\lambda_j}) = \rho_i(\prod_{j=1}^{m_i} f_{\alpha_{ij}}^{\lambda_j}) .$$

Definition 5.6. An admissible set $(\mathbf{R}, \rho_i, \Lambda_i), i = 1, 2$ of near weight structures on $\mathbf{R}$ is said to be of finite type if there exist $f_{\alpha_{ij}} \in \mathbf{R}$, with $i = 1, 2; j = 1, \dots, m_i$, such that

$$\rho_i(\prod_{j=1}^{m_i} f_{\alpha_{ij}}^{\lambda_j}) = \sum_{j=1}^{m_i} \lambda_j \alpha_{ij} = \gamma \in \Lambda_i \quad \text{and} \quad \rho_\ell(\prod_{j=1}^{m_i} f_{\alpha_{ij}}^{\lambda_j}) = x_l(\gamma) \in \Lambda_\ell ,$$

$\ell \in \{1, 2\}$ with $\ell \neq i$.

Example 5.7. The admissible set of near weight structures in Example 4.4 is of finite type.

Proposition 5.8. *Let $(\mathbf{R}, \rho_i, \Lambda_i)$, $i = 1, 2$ be an admissible set of near weight structures of finite type on $\mathbf{R}$. Then $\mathbf{R}$ is a finitely generated algebra over $\mathbb{F}$.*

Proof. We will show that $\mathbf{R} = \mathbb{F}[\{f_a : a \in \bar{\Omega}\}]$, where $\bar{\Omega}$ is defined in (5.2). Let $f \in \mathbf{R} \setminus \{0\}$. Suppose that $\rho_2(f) = 0$; we will proceed by induction on $\rho_1(f) = \gamma$. If $\gamma = 0$, $f \in \mathcal{U}_{\rho_1} \cap \mathcal{U}_{\rho_2} = \mathbb{F}$ and hence $f \in \mathbb{F}[\{f_a : a \in \bar{\Omega}\}]$. If $\gamma \succ_1 0$, by induction for all $g \in \mathbf{R}$ such that $\rho_1(g) \prec_1 \gamma$ and $\rho_2(g) = 0$ we get $g \in \mathbb{F}[\{f_a : a \in \bar{\Omega}\}]$. As $\rho_1(f) = \gamma$ we get

$$\rho_1(f) = \gamma = \sum_{j=1}^{m_1} \lambda_j \alpha_{1j} = \rho_1(\prod_{j=1}^{m_1} f_{\alpha_{1j}}^{\lambda_j}) .$$

From Axiom (N4) in Definition 3.1 there exist $\lambda \in \mathbb{F}^*$ such that $\rho_1(f - \lambda \prod_{j=1}^{m_1} f_{\alpha_{1j}}^{\lambda_j}) \prec_1 \gamma$. But $\rho_2(\prod_{j=1}^{m_1} f_{\alpha_{1j}}^{\lambda_j}) = x_2(\gamma) = 0$ as $\gamma \in \mathcal{H}_{\rho_1}$, and hence $\rho_2(f - \lambda \prod_{j=1}^{m_1} f_{\alpha_{1j}}^{\lambda_j}) \preceq_2 0$. Therefore, by induction $f - \lambda \prod_{j=1}^{m_1} f_{\alpha_{1j}}^{\lambda_j} \in \mathbb{F}[\{f_a : a \in \bar{\Omega}\}]$ and so f is so.

Now suppose that $\rho_2(f) = \beta \neq 0$; then there exists $\mu_{j1} \in \mathbb{N}_0$ such that

$$\rho_2(f) = \beta = \sum_{j=1}^{m_2} \mu_{j1} \alpha_{2j} = \rho_2(\prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j1}}) .$$

From the aforementioned Axiom (N4) there exist $\lambda_1 \in \mathbb{F}^*$ such that $\rho_2(f - \lambda_1 \prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j1}}) \prec_2 \beta$. Moreover, there exist $\mu_{j2} \in \mathbb{N}_0$ such that

$$\rho_2(f - \lambda_1 \prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j1}}) = \sum_{j=1}^{m_2} \mu_{j2} \alpha_{2j} = \rho_2(\prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j2}}) ,$$

and from Axiom (N4) again there exist $\lambda_2 \in \mathbb{F}^*$ such that

$$\rho_2(f - \lambda_1 \prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j1}} - \lambda_2 \prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j2}}) \prec_2 \rho_2(\prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{j2}}) \, .$$

By continuing this process, as Λ_2 is well ordered, there exist $\lambda_i \in \mathbb{F}$, $\mu_{ji} \in \mathbb{N}_0$ such that

$$\rho_2(f - \sum_i \lambda_i \prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{ji}}) = 0 \, .$$

Therefore from the first part of the proof of the proposition it follows that $f - \sum_i \lambda_i (\prod_{j=1}^{m_2} f_{\alpha_{2j}}^{\mu_{ji}}) \in \mathbb{F}[\{f_a : a \in \bar{\Omega}\}]$ and so f is so. □

A consequence of Propositions 4.5 and 5.8 above is that an $\mathbb{F}$-algebra $\mathbf{R}$ admitting two admissible near weight structures is in fact a finitely generated domain over $\mathbb{F}$; thus it is $\mathbb{F}$-isomorphic to an affine $\mathbb{F}$-algebra, says

$$\mathbf{R} \cong \mathbb{F}[X_1, \ldots, X_n]/I \, ,$$

where I is a prime ideal of $\mathbb{F}[X_1, \ldots, X_n]$. Let $\mathbf{K} = \mathbf{K}(\mathbf{R})$ be the field of quotients of $\mathbf{R}$. Thus the Krull dimension of $\mathbf{R}$ is equal to the transcendence degree of $\mathbf{K}$ over $\mathbb{F}$ (see e.g. [6, Ch. 8, Thm. A]).

Definition 5.9. Let Λ be a (commutative) semigroup. The rational rank of Λ is defined by

$$\text{rat.rank}(\Lambda) := \dim_{\mathbb{Q}}(G(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Q}) \, ,$$

where $G(\Lambda)$ is the group of differences of Λ (cf. [8, Def. 5.3], [20, §1]).

Proposition 5.10. *Let $\mathbf{R}$ be an $\mathbb{F}$-algebra and $(\mathbf{R}, \rho_i, \Lambda_i)$, $i = 1, 2$ be an admissible near weight structures of finite type on $\mathbf{R}$. Let $\mathbf{K}$ be the quotient field of $\mathbf{R}$. Then the transcendence degree of $\mathbf{K}$ over $\mathbb{F}$ is at least the rational rank of Λ_i, $i = 1, 2$.*

Proof. Similar to the proof of [8, Prop. 11.3]. For the sake of completeness we write a proof. Let $r_i := \text{rat.rank}(\Lambda_i)$. Then we can find r_i rationally independent elements $\gamma_1, \ldots, \gamma_{r_i} \in \Lambda_i$ (see [20, §1]). Choose $f_{\gamma_j} \in \mathbf{R}$ such that $\rho(f_{\gamma_j}) = \gamma_j$, $j = 1, \ldots, r_i$. Then $f_{\gamma_1}, \ldots, f_{\gamma_{r_i}}$ are algebraically independent over $\mathbb{F}$; otherwise there would exist $g \in \mathbb{F}[X_1, \ldots, X_{r_i}]$, $g \neq 0$, such that $g(f_{\gamma_1}, \ldots, f_{\gamma_{r_i}}) = 0$. So there exist two distinct terms $\lambda X_1^{\alpha_1} \ldots X_{r_i}^{\alpha_{r_i}}$ and $\mu X_1^{\beta_1} \ldots X_{r_i}^{\beta_{r_i}}$ of $g(X_1, \ldots, X_{r_i})$ with $\alpha_j, \beta_j \in \mathbb{N}_0$ and $\lambda, \mu \in \mathbb{F}$ such that

$$\rho(\lambda f_{\gamma_1}^{\alpha_1} \ldots f_{\gamma_{r_i}}^{\alpha_{r_i}}) = \rho(\mu f_{\gamma_1}^{\beta_1} \ldots f_{\gamma_{r_i}}^{\beta_{r_i}}) \, .$$

From the axioms in Definition 3.1, $\sum_{j=1}^{r_i} (\alpha_j - \beta_j) \rho(f_{\gamma_j}) = 0$; i.e., $\sum_{j=1}^{r_i} (\alpha_j - \beta_j) \gamma_j = 0$. As $(\alpha_i - \beta_i)$ are not all zero, then $\gamma_1, \ldots, \gamma_{r_i}$ are rationally dependent, a contradiction. Then

$S := \mathbb{F}[f_{\gamma_1}, \ldots, f_{\gamma_{r_i}}]$ is a subanel of $\mathbf{R}$ and it is isomorphic to $\mathbb{F}[X_1, \ldots, X_{r_i}]$. Therefore

$$r_i = \dim_{\text{Krull}}(S) \leq \dim_{\text{Krull}}\mathbf{R} = \text{tr.deg}(\mathbf{K}|\mathbb{F})\,.$$

□

Next, in a natural way, we associate a valuation to a near weight function on $\mathbf{K}$; to this end, the following two auxiliary results are quite useful.

Lemma 5.11. *Let $i \in \{1,2\}$ and $f \in \mathcal{M}_{\rho_i}$. If $g \in \mathcal{U}_{\rho_i}\backslash\mathbb{F}$, there exist $\lambda \in \mathbb{F}$ such that $\rho_i(f(g-\lambda)) \prec_i \rho_i(f)$.*

Proof. Similar to the proof of [14, Lemma 2.3]. □

Lemma 5.12. *Let $f \in \mathbf{R}\backslash\{0\}$ and $i \in \{1,2\}$. Then there exists $g \in \mathcal{M}_{\rho_i}$ such that $fg \in \mathcal{M}_{\rho_i}$.*

Proof. Let $f \in \mathbf{R}\backslash\{0\}$. If $f \in \mathbb{F}$, for any $g \in \mathcal{M}_{\rho_i}$ $fg \in \mathcal{M}_{\rho_i}$. Suppose now $f \in \mathbf{R}\backslash\mathbb{F}$. If $\rho_i(f) \succ_i 0$, then for any $g \in \mathcal{M}_{\rho_i}$ $\rho_i(fg) = \rho_i(f) + \rho_i(g) \succ_i 0$, and hence $fg \in \mathcal{M}_{\rho_i}$. Now, suppose $f \in \mathcal{U}_{\rho_i}\backslash\mathbb{F}$. Then $\rho_j(f) \succ_j 0$ with $j = 1,2$ and $j \neq i$. Let us show that there exists $g \in \mathcal{U}_{\rho_j}\backslash\mathbb{F}$ such that $\rho_j(fg) = 0$. Let $h \in \mathcal{U}_{\rho_j}\backslash\mathbb{F}$. Then $\rho_j(fh) \preceq_j \rho_j(f)$ and from Lemma 5.11 there exists $\lambda_1 \in \mathbb{F}$ such that $\rho_j(f(h-\lambda_1)) \prec_j \rho_j(f)$. If $\rho_j(f(h-\lambda_1)) = 0$, take $g = h - \lambda_1$; otherwise, if $\rho_j(f(h-\lambda_1)) \succ_j 0$ then $\rho_j(f(h-\lambda_1)h) \preceq_j \rho_j(f(h-\lambda_1))$. From Lemma 5.11, there exist $\lambda_2 \in \mathbb{F}$ such that $\rho_j(f(h-\lambda_1)(h-\lambda_2)) \prec_j \rho_j(f(h-\lambda_1))$. Continuing this process, and from the fact that Λ_j is well ordered, we can find $g = f \cdot \prod_{finite}(h-\lambda_k) \in \mathcal{U}_{\rho_j}\backslash\mathbb{F}$ such that $\rho_j(fg) = 0$. Hence $g \in \mathcal{M}_{\rho_i}$, and $\rho_i(fg) \succeq_i 0$. If $\rho_i(fg) \succ_i 0$, $fg \in \mathcal{M}_{\rho_i}$. If $\rho_i(fg) = 0$, $fg \in \mathcal{U}_{\rho_1} \cap \mathcal{U}_{\rho_2} = \mathbb{F}$ and therefore $fg^2 \in \mathcal{M}_{\rho_i}$. □

From the above lemma we see that if $f, g \in \mathbf{R}\backslash\{0\}$, there exist $h, z \in \mathcal{M}_{\rho_i}$, $i = 1,2$, such that $fh, gz \in \mathcal{M}_{\rho_i}$ and hence $f(hz), g(hz) \in \mathcal{M}_{\rho_i}$. Thus we can define a map $\nu_i : \mathbf{K} \to G(\Lambda_i) \cup \{\infty\}$ by $\nu_i(0) := \infty$ and

$$\nu_i(f/g) := \rho_i(hg) - \rho_i(hf)\,, \tag{5.3}$$

where $f, g \in \mathbf{R}\backslash\{0\}$ and $h \in \mathcal{M}_{\rho_i}$ is such that $fh, gh \in \mathcal{M}_{\rho_i}$ for $i = 1,2$. This map is a valuation in the sense of Section 2, with $\mathbf{R}_{\nu_i} = \{f \in \mathbf{K} : \nu_i(f) \succeq_i 0\}$ as its valuation ring, $\mathbf{M}_{\nu_i} = \{f \in \mathbf{K} : \nu_i(f) \succ_i 0\}$ as its maximal ideal, and with residue field isomorphic to $\mathbb{F}$; cf. [4, Lemma 3.17], [15, Lemmas 5.11, 5.13].

Note that each near weight function $\rho_i : \mathbf{R} \to \Lambda_i \cup \{-\infty\}$ can be recovered from ν_i (cf. Example 3.5) as follows:

$$\rho_i(f) = \begin{cases} -\infty & \text{if } f = 0\,, \\ 0 & \text{if } 0 \preceq \nu_i(f)\,, \\ -\nu_i(f) & \text{if } \nu_i(f) \prec 0\,; \end{cases}$$

Next we will characterize certain $\mathbb{F}$-algebras $\mathbf{R}$ which admit two admissible near weight structures.

Theorem 5.13. *Let $\mathbf{R}$ be an $\mathbb{F}$-algebra, let $(\mathbf{R}, \rho_i, \Lambda_i)$, $i = 1, 2$ be two different admissible near weight structures of finite type on $\mathbf{R}$. Suppose that each group of difference $G(\Lambda_i)$ has an isolated subgroup with rational rank* $\mathrm{rat.rank}(\Lambda_i) - 1$*. Suppose further that* $\mathrm{rat.rank}(\Lambda_i) = \mathrm{tr.deg}(\mathbf{K}|\mathbb{F})$*, for each i, where $\mathbf{K}$ is the quotient field of $\mathbf{R}$. Then $\mathbf{K}$ is an algebraic function field of* $\mathrm{rat.rank}(\Lambda_i)$ *independent variables over $\mathbb{F}$, and the integral closure $\bar{\mathbf{R}}$ of $\mathbf{R}$ in $\mathbf{K}$ is a subring of $\mathbf{K}$ consisting of regular functions with poles in at least two prime divisors of $\mathbf{K}$.*

Proof. From Proposition 5.8 and Theorem A in [6, Ch. 8], $\mathbf{K}$ is an algebraic function field of transcendence degree $r := \mathrm{rat.rank}(\Lambda_i)$ over $\mathbb{F}$. Let ν_i be the Krull valuation of $\mathbf{K}$ associated to ρ_i (as in (5.3)), let $\mathbf{R}_i = \mathbf{R}_{\nu_i}$ be the valuation ring of ν_i, let $\mathbf{M}_i$ be its maximal ideal, and κ_i be its residue field. From the remark after the proof of Lemma 5.12, $\kappa_i = \mathbf{R}_i/\mathbf{M}_i \cong \mathbb{F}$ and thus $\dim(\nu_i) = \mathrm{tr.deg}(\kappa_i|\mathbb{F}) = 0$. Let Δ_i be the isolated subgroup of $G(\Lambda_i)$ with $\mathrm{rat.rank}(\Delta_i) = r - 1$. Then, from Valuation Theory, we have that $\nu_i = \mu_i \circ \bar{\nu}_i$, where each $\mu_i : \mathbf{K} \to (G(\Lambda_i)/\Delta_i) \cup \{\infty\}$ is a discrete valuation with rank 1 in $\mathbf{K}$, because $\mathrm{rank}(\mu_i) \le \mathrm{rat.rank}(\mu_i) = 1$, and $\bar{\nu}_i : \kappa_{\mu_i} \to \Delta_i \cup \{\infty\}$ is a Krull valuation of residue field κ_{μ_i} of μ_i. As $\mathrm{rat.rank}(\mu_i) + \dim(\mu_i) \le \mathrm{tr.deg}(\mathbf{K}|\mathbb{F}) = r$, $\dim(\mu_i) \le r - 1$. But as $\mathrm{rat.rank}(\bar{\nu}_i) = \mathrm{rat.rank}(\Delta_i) = r - 1$ and the residue field $\kappa_{\bar{\nu}_i}$ of $\bar{\nu}_i$ is equal to the residue field of ν_i (see [20, Prop. 1.12]) we get

$$r - 1 = \mathrm{rank}(\bar{\nu}_i) + \dim(\bar{\nu}_i) \le \mathrm{tr.deg}(\kappa_{\mu_i}|\mathbb{F}) = \dim(\mu_i) \le r - 1\,.$$

Then $\dim(\mu_i) = r - 1$ and hence each μ_i is a prime divisor of $\mathbf{K}|\mathbb{F}$.

Let $\bar{\mathbf{R}}$ be the integral closure of $\mathbf{R}$ in $\mathbf{K}$. Let $S(\mathbf{R}) := \{\omega \text{ prime divisor in } \mathbf{K}|\mathbb{F} : \mathbf{R} \subseteq \mathbf{R}_\omega\}$. Then $\bar{\mathbf{R}} = \cap_{\omega \in S(\mathbf{R})} \mathbf{R}_\omega$. We will show that $\mu_i \notin S(\mathbf{R})$ for $i = 1, 2$. In fact, suppose $\mathbf{R} \subseteq R_{\mu_i}$, $i = 1, 2$. Then for any $f \in \mathbf{M}_{\rho_i}$, $\rho_i(f) \succ_i 0$, or else $\nu_i(f) \prec_i 0$ and hence $\mu_i(f) \le 0$. As $\mathbf{R} \subseteq \mathbf{R}_{\mu_i}$, we have $\mu_i(f) = 0$ for all $f \in \mathcal{M}_{\rho_i}$. Let $a/b \in \mathbf{K}$ with $a, b \in \mathbf{R}\setminus\{0\}$ such that $\mu_i(a/b) > 0$. From Lemma 5.12, there exists $g \in \mathcal{M}_{\rho_i}$ such that $ga, gb \in \mathcal{M}_{\rho_i}$. Hence $0 < \mu_i(a/b) = \mu_i(ga/gb) = \mu_i(ga) - \mu_i(gb) = 0$, a contradiction. □

Corollary 5.14. *Notation as above. If for any $f \in \mathbf{R}\backslash\mathbb{F}$ there exists $i \in \{1,2\}$ such that $\mu_i(f) < 0$, then the integral closure $\bar{\mathbf{R}}$ of $\mathbf{R}$ in $\mathbf{K}$ is a subring of $\mathbf{K}$ consisting of regular functions with poles only at two prime divisors of $\mathbf{K}$.*

Proof. As was noticed in the proof of Theorem 5.13, $\bar{\mathbf{R}} = \cap_{\omega\in S(\mathbf{R})} R_\omega$, and $\mu_1, \mu_2 \notin S(\mathbf{R})$. Let S be the set of all prime divisors of $\mathbf{K}|\mathbb{F}$. We will show that $S(\mathbf{R}) = S\backslash\{\mu_1,\mu_2\}$. Suppose that $S(\mathbf{R}) \cup \{\mu_1,\mu_2\} \neq S$. Let

$$\mathbf{R}' = \cap_{\omega\in S(\mathbf{R})\cup\{\mu_1,\mu_2\}}\mathbf{R}_\omega \subseteq \bar{\mathbf{R}}\,.$$

Let $x \in \mathbf{R}'$ such that $\mu_i(x) > 0$ for $i = 1,2$ (the existence of this element is given e.g. in [2, Ch. VII, §1.5, Prop. 9]). Let $I = \{y \in \mathbf{R} : y\bar{\mathbf{R}} \subset \mathbf{R}\} \neq (0)$ be the conductor of $\mathbf{R}$ in $\bar{\mathbf{R}}$. Then for any $y \in I$, $yx \in \mathbf{R}$; so $\mu_i(xy) < 0$ for some i; i.e., $\mu_i(x) < \mu_i(y^{-1})$. But μ_i is Archimedean, and so there exists a positive integer n_i such that $n_i\mu_i(x) > \mu_i(y^{-1})$; i.e., $\mu_i(x^{n_i}y) > 0$. For $j \in \{1,2\}$,$j \neq i$, we have $\mu_j(xy) < 0$ or $\mu_j(xy) \geq 0$. If $\mu_j(xy) < 0$, there exists a positive integer n_j such that $\mu_j(x^{n_j}y) > 0$ as μ_j is Archimedean. If $\mu_j(xy) \geq 0$, for any positive integer $n > 1$ we get that $\mu_j(x^n y) > 0$, because $\mu_j(x) > 0$. So, let $k = max\{n_i : i = 1,2\} \geq 1$. Then $\mu_i(x^m y) > 0$ for all i and for all $m \geq k$. Hence $x^m y \in \mathcal{U}_{\rho_i}$ for all $i \in \{1,2\}$ and for all $m \geq k$, because $\nu_i(x^m y) \succeq 0$. Thus $x^m y \in \mathcal{U}_{\rho_1} \cap \mathcal{U}_{\rho_2} = \mathbb{F}$ for all $m \geq k$, a contradiction. Therefore $\bar{\mathbf{R}} = \cap_{\omega\in S\backslash\{\mu_1,\mu_2\}}\mathbf{R}_\omega$ □

Proposition 5.15. *Let $(\mathbf{R},\rho_i,\Lambda_i), i = 1,2$ be two admissible near weight structures of finite type on $\mathbf{R}$ such that each group of difference $G(\Lambda_i)$ has an isolated subgroup with rational rank* rat.rank$(\Lambda_i) - 1$. *Suppose further that* rat.rank$(\Lambda_i) =$ tr.deg$(\mathbf{K}|\mathbb{F})$, *for each i, where $\mathbf{K}$ is the quotient field of $\mathbf{R}$. Then $\mathbf{R}$ is the ring of regular functions of an affine algebraic variety, whose projective closure $\mathcal{X}$ have two irreducible divisors Z_1 and Z_2 at infinity. Furthermore, if for any $f \in \mathbf{R}\backslash\mathbb{F}$, there exists $i \in \{1,2\}$ such that $\mu_i(f) < 0$, where μ_i is the prime divisor of $\mathbf{K}|\mathbb{F}$ on the decomposition of ν_i. The normalization of $\mathcal{X}$ have only two irreducible divisors Z_1 and Z_2 at infinity.*

Proof. Let $\mathcal{X}$ be the projective algebraic variety defined by $\mathbf{R}$ and let $\bar{\mathcal{X}}$ be its normalization. Then from [20, Prop. 2.3] and Theorem 5.13 each valuation μ_i, $i = 1,2$, is centering at an irreducible divisor Z_i of $\bar{\mathcal{X}}$. Now, if for any $f \in \mathbf{R}\backslash\mathbb{F}$ such that $\mu_i(f) < 0$ for some i, then from Corollary 5.14 it follows that Z_1 and Z_2 are the only irreducible divisors of $\bar{\mathcal{X}}$ at infinity. □

Remark 5.16. From [20, Props. 2.3, 2.4] we have that each prime divisor μ_i on the decomposition $\nu_i = \mu_i \circ \bar{\nu}_i$ has its center in a subvariety D_i of $\mathcal{X}$ such that $\dim(D_i) \leq \dim(\mu_i)$; moreover each valuation ν_i is centered at a rational point $Q_i \in D_i$.

When $\Lambda_i = \mathbb{N}_0$ for $i = 1, 2$, we have that $\mathcal{X}$ in Proposition 5.15 is an affine algebraic curve, and the irreducible divisors Z_1 e Z_2 are rational points of $\bar{\mathcal{X}}$. Now, suppose that $\mathbb{F}$ is a finite field. A Similar proof as in [4, Thm. 3.21] gives the following result.

Theorem 5.17. *Let* $(\mathbf{R}, \rho_i, \Lambda_i)$, $i = 1, 2$ *be two finite type admissible set of near weight structures on* $\mathbf{R}$*, let* $\varphi : \mathbf{R} \to \mathbb{F}^n$ *be a epimorphism of* $\mathbb{F}$*-algebras and* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$*. Then* $E(\alpha)$ *is a Goppa code* $C_{\mathcal{L}}(D, G)$ *with* $G = \alpha_1 Z_1 + \alpha_2 Z_2$*.*

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