BIRTH OF LIMIT CYCLES BIFURCATING FROM A NONSMOOTH CENTER

CLAUDIO A. BUZZI[1], TIAGO DE CARVALHO[2] AND MARCO A. TEIXEIRA[3]

Abstract. This paper is concerned with a codimension analysis of a two-fold singularity of piecewise smooth planar vector fields, when it behaves itself like a center of smooth vector fields (also called nondegenerate Σ-center). We prove that any nondegenerate Σ-center is Σ-equivalent to a particular normal form Z_0. Given a positive integer number k we explicitly construct families of piecewise smooth vector fields emerging from Z_0 that have k hyperbolic limit cycles bifurcating from the nondegenerate Σ-center of Z_0 (the same holds for $k = \infty$). Moreover, we also exhibit families of piecewise smooth vector fields of codimension k emerging from Z_0. As a consequence we prove that Z_0 has infinite codimension.

1. Introduction

One of the most difficult problems when we study piecewise smooth vector fields (PSVFs for short) is to determine the real codimension of the singularities involved. Several recent papers [2, 3, 14, 15] describe the dynamics around typical singularities of certain parameterized families of planar PSVFs by means of the analysis of the correspondent bifurcation diagram. On the other hand very little has been done for the knowledge of all possible topological types that can arise when some specific degenerate vector field, having a simple phase portrait, is perturbed. We can draw a parallel situation with the smooth universe following the theory developed by F. Takens [16] and that is known today under the name of the *Hopf-Takens bifurcation*. This is an infinite-codimension bifurcation, more precisely, the successive vanishing coefficients of the normal form displays possibility of birth of k limit cycles near the center, for any k. A more recent version of Takens theory was proposed in [4, 5] and has been pushed by the same time very far in the analytic, complex and real, polynomial case. See also in [12] a study for perturbed *Hamiltonian systems*. We find many models of PSVFs [8, 10], whose phase portraits are topologically equivalent to a smooth center and it would not be possible, a priori, to know the real codimension of the system. In this paper we prove that a normal form of the so called two-fold singularity analogous to a center has infinite codimension.

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One of our main motivations comes from the remarkable work of I. Ekeland in [10] where the main problem in the classical calculus of variations was carried out by means of the classification of generic typical singularities in the world of planar refracted Hamiltonian vector fields. There is a class of such typical singularities where their dynamics are illustrated in Figure 1.

Another motivation comes from mechanical models like the nonsmooth harmonic oscillator (see Example 3 in [8]). Consider a unit mass subject to a discontinuous spring force. The spring does not exert any force when the mass is at the reference position $z = 0$. When the mass is displaced to the right, that is, $z > 0$, the spring exerts a constant negative force that pulls it back to the reference position. When the mass is displaced to the left, that is, $z < 0$, the spring exerts a constant positive force that pulls it back to the reference position. According to Newton's second law (see [7]), the system evolution is described by

$$\ddot{z} + 2sign(z) = 0.$$

By defining the state variables $x = \dot{z}/2$ and $y = z$, the equation can be rewritten as our main object of study in this work: the PSVF Z_0 given by (1).

Roughly speaking a PSVF $Z = (X, Y)$ on the plane is a pair of C^r-vector fields X and Y, where X and Y are restricted to regions of the plane separated by a smooth curve Σ. In Section 2 we give a precise definition. Let Ω be the set of all PSVF endowed with the C^r product topology. We say that $p \in \Sigma$ is a fold point for the vector field X if the trajectory of X passing thought p has a quadratic contact with Σ. The fold point can be visible or invisible. It is visible if the trajectory passing though p remains in the same side where X is defined, otherwise it is invisible. We say that p is a *two-fold singularity* for $Z = (X, Y) \in \Omega$ if p is a fold point for both X and Y. In this paper, topologically speaking, we consider the case when the two-fold singularity behaves itself like a center of smooth vector fields (see Figure 1). We call it *nondegenerate Σ-center* (see Definition 2).

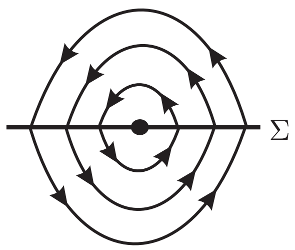

FIGURE 1. Nondegenerate Σ-center.

Some extremely relevant questions in this context is to determine how many limit cycles can bifurcate from this nondegenerate Σ-center and what is the codimension of this singularity. Our first result ensures that any nondegenerate Σ-center is Σ-*equivalent* (see Definition 3) to its normal form.

Now we state the main results in the paper.

Proposition 1. *Let $Z = (X, Y)$ be a PSVF presenting a nondegenerate Σ-center, then Z is Σ-equivalent to*

$$(1)\qquad Z_0(x,y) = \begin{cases} X_0(x,y) = \begin{pmatrix} -1 \\ 2x \end{pmatrix} & \text{if } y \geq 0, \\ Y_0(x,y) = \begin{pmatrix} 1 \\ 2x \end{pmatrix} & \text{if } y \leq 0. \end{cases}$$

Theorem A. *Let Z_0 be given by* (1)*. For any neighborhood $\mathcal{W} \subset \Omega$ of Z_0 and for any integer $k > 0$, there exists $\widetilde{Z} \in \mathcal{W}$ such that $\widetilde{Z}$ has k hyperbolic limit cycles. The same holds if $k = \infty$.*

Theorem B. *Let Z_0 be given by* (1)*. For any neighborhood $\mathcal{W} \subset \Omega$ of Z_0 and for any integer $k > 0$, there exists $Z_\varepsilon^k \in \mathcal{W}$ of codimension k.*

Moreover, in the proof of Theorem A, for a given positive integer k we explicitly build families of PSVFs bifurcating exactly k hyperbolic limit cycles from the nondegenerate Σ-center. And an immediate consequence of Theorem B is the following.

Theorem C. *The PSVF Z_0, given in* (1) *and presenting a nondegenerate Σ-center, has infinite codimension.*

The paper is organized as follows. In Section 2 we introduce the terminology, some definitions and the basic theory about PSVFs. Sections 3 and 4 are devoted to prove Proposition 1 and Theorem A. In Section 5 we build a family Z_ε^k such that the map that gives its first return to Σ has codimension k. Based on it we prove Theorem B and Theorem C. Besides, we wrote a section called Appendix were we pave the way in order to prove the results in Section 5.

2. Preliminaries

Let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^2$. We consider a codimension one manifold Σ of $\mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. In this paper we assume that $\Sigma = f^{-1}(0)$, where $f(x, y) = y$.

Designate by χ the space of C^r-vector fields on $V \subset \mathbb{R}^2$ endowed with the C^r-topology, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^2$ such that

$$(2)\qquad Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases}$$

where $X = (f_1, g_1), Y = (f_2, g_2) \in \chi$. We endow Ω with the product topology. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we will accept it to be multi-valued in points of Σ. The basic results of differential equations, in this context, were stated by Filippov in [11].

Consider the notation $X.f(p) = \langle \nabla f(p), X(p)\rangle$ and, for $i \geq 2$, $X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p)\right\rangle$, where $\langle .,.\rangle$ is the usual inner product in $\mathbb{R}^2$. We say that a point $p \in \Sigma$ is a Σ-*fold point* of X if $X.f(p) = 0$ but $X^2.f(p) \neq 0$. Moreover, $p \in \Sigma$ is a *visible* (respectively *invisible*) Σ-fold point of X if $X.f(p) = 0$ and $X^2.f(p) > 0$ (respectively $X^2.f(p) < 0$). We say that $p \in \Sigma$ is a *two-fold singularity* of Z if p is a Σ-fold point for both X and Y.

Consider the case when the PSVF $Z = (X, Y)$ has q as an invisible fold point for both X and Y. In polar coordinates $(x, y) = (r\cos\theta,\ r\sin\theta)$, Z is such that each solution starting at a point $(r_1, 0) \in \Sigma$ flows into Σ^+ and comes back to Σ at a point (r_2, π); and each solution starting at a point $(r_3, \pi) \in \Sigma$ flows into Σ^- and comes back to Σ at a point $(r_4, 2\pi)$. Let $r^+(\theta, \rho)$ (resp. $r^-(\theta, \rho)$) be the solution of X (resp. Y) such that $r^+(0, \rho) = \rho$ (resp. $r^-(\pi, \rho) = \rho$). Then (see Figure 2), we can define the *positive half-return map* as $\Pi_X(\rho) = r^+(\pi, \rho)$, and the *negative half-return map* as $\Pi_Y(\rho) = r^-(2\pi, \rho)$. The complete *return map* associated to Z is given by the composition of these two maps

$$\Pi_Z(\rho) = \Pi_Y(\Pi_X(\rho)). \tag{3}$$

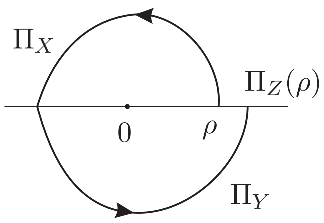

FIGURE 2. Return map of $Z = (X, Y)$.

In next definition we consider the case when the two-fold singularity behaves itself like a center of smooth vector fields.

Definition 2. *Consider $Z \in \Omega$. We say that $q \in \Sigma$ is a* ***nondegenerate Σ-center*** *of Z if q is an invisible fold point for both X and Y, and there is a neighborhood $U \subset \mathbb{R}^2$ of q filled up with a one-parameter family γ_s of closed orbits of Z in such a way that the orientation is preserved (see Figure 1).*

Another important definition is the concept of equivalence between two PSVFs.

Definition 3. *Two PSVFs $Z = (X, Y)$, $\widetilde{Z} = (\widetilde{X}, \widetilde{Y}) \in \Omega$ defined in open sets $U, \widetilde{U}$ and with switching manifold Σ are* ***Σ-equivalent*** *if there exists* *an orientation preserving homeomorphism* $h : U \to \widetilde{U}$ *that sends* $U \cap \Sigma$ *to* $\widetilde{U} \cap \Sigma$*, the orbits of* X *restricted to* $U \cap \Sigma^+$ *to the orbits of* $\widetilde{X}$ *restricted to* $\widetilde{U} \cap \Sigma^+$*, and the orbits of* Y *restricted to* $U \cap \Sigma^-$ *to the orbits of* $\widetilde{Y}$ *restricted to* $\widetilde{U} \cap \Sigma^-$.

Remark 1.

- *In this work we decide consider only perturbations of* (1) *that keep the origin as a two-fold singularity. This assumption is important because in this case the return map is always well defined.*
- *In this sense the return map of all trajectories considered in this paper is given by the composition of two involutions (see* [17]*).*

3. Proof of Proposition 1

In this section we construct a Σ-preserving homeomorphism h that sends orbits of $Z = (X, Y)$, presenting a nondegenerate Σ-center, to orbits of $Z_0 = (X_0, Y_0)$ given by (1). Let $\overline{p}$ (respectively, p) be the two-fold singularity of Z (respectively, Z_0) (see Figure 3).

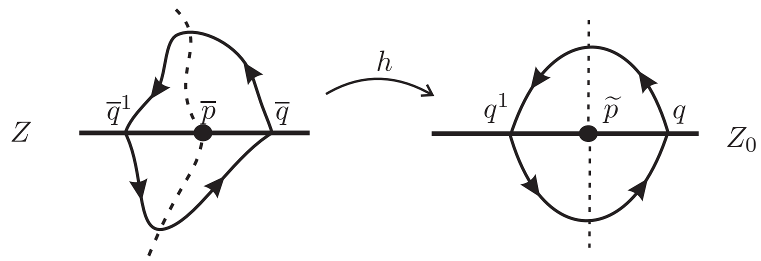

Figure 3. Construction of the homeomorphism.

Identify $\overline{p}$ with p, i.e., $h(\overline{p}) = p$. Without loss of generality consider that orbits of Z are oriented in an anti-clockwise sense. By arc length parametrization identify $\overline{\Sigma}(+)$ with $\Sigma(+)$, where $\overline{\Sigma}(+)$ (respectively, $\Sigma(+)$) is the set of all points of Σ situated on the right of $\overline{p}$ (respectively, p). So, $h(\overline{\Sigma}(+)) = \Sigma(+)$. Consider a point $\overline{q} \in \overline{\Sigma}(+)$ (respectively, $h(\overline{q}) = q \in \Sigma(+)$). Since $\overline{p}$ (respectively, p) is an invisible Σ-fold point of X (respectively, X_0) by the Implicit Function Theorem (abbreviated by IFT), there exists a smallest time $\overline{t}_1 > 0$ (respectively, $t_1 > 0$), depending on $\overline{q}$ (respectively, q), such that $\phi_X(\overline{q}, \overline{t}_1) := \overline{q}^1 \in \overline{\Sigma}(-)$ (respectively, $\phi_{X_0}(q, t_1) := q^1 \in \Sigma(-)$), where $\overline{\Sigma}(-)$ (respectively, $\Sigma(-)$) is the set of all points of Σ situated on the left of $\overline{p}$ (respectively, p) and ϕ_W denotes the flow of the vector field W. Identify the orbit arcs $\gamma_{\overline{q}}^{\overline{q}^1}(X)$ and $\gamma_q^{q^1}(X_0)$ of X and X_0 with initial points $\overline{q}$ and q and final points $\overline{q}^1$ and q^1, respectively, by arc length parametrization. Now, since Z (respectively, Z_0) presents a nondegenerate Σ-center and $\overline{p}$

(respectively, p) is an invisible Σ-fold point of Y (respectively, Y_0) by the IFT, there exists a smallest time $\overline{t}_2 > 0$ (respectively, $t_2 > 0$), depending on $\overline{q}^1$ (respectively, q^1), such that $\phi_Y(\overline{q}^1, \overline{t}_2) := \overline{q} \in \overline{\Sigma}(+)$ (respectively, $\phi_{Y_0}(q^1, t_2) := q \in \Sigma(+)$). Identify the orbit arcs $\gamma_{\overline{q}^1}^{\overline{q}}(Y)$ and $\gamma_{q^1}^{q}(Y_0)$ of Y and Y_0 with initial points $\overline{q}^1$ and q^1 and final points $\overline{q}$ and q, respectively, by arc length parametrization. Repeat this procedure for all points in $\overline{\Sigma}(+)$ and get that $h(\overline{\Sigma}^+) = \Sigma^+$, $h(\overline{\Sigma}(-)) = \Sigma(-)$ and $h(\overline{\Sigma}^-) = \Sigma^-$. Therefore the homeomorphism h preserves Σ and sends orbits of Z to orbits of Z_0.

4. Proof of Theorem A

In order to obtain Theorem A we need some lemmas. Observe that both vector fields X_0 and Y_0 in the normal form (1) are written as $W(x,y) = (\pm 1, g(x))$ (particularly, $g(x) = 2x$ in such expression). Next lemma gives how are the trajectories of such systems.

Lemma 4. *The trajectories of a vector field $W(x,y) = (1, g(x))$ in χ are obtained by vertical translations of the graph of $G(x)$, where $\frac{\partial}{\partial x}G(x) = g(x)$.*

Proof. Since $W(x,y) = (\dot{x}, \dot{y}) = (1, g(x)) \in \chi^r$ we obtain that

$$x(t) = t + c \text{ and } y(t) = \int g(t+c)dt = G(t+c) + K$$

where c, $K \in \mathbb{R}$ and G is a primitive of g. Now, take $u = t + c$ and the trajectories of $W(x,y)$ are given by $(u, G(u) + K)$ which are vertical translations of the graph of $G(u)$. □

Observe that if the trajectories of a vector field X is known then the trajectories of $-X$ is also known.

In what follows, $h : \mathbb{R} \to \mathbb{R}$ will denote the C^∞-function given by

$$h(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ e^{-1/x}, & \text{if } x > 0. \end{cases}$$

Lemma 5. *Consider the function $\xi_\varepsilon^f(x) = \varepsilon h(x)(\varepsilon - x)(2\varepsilon - x)\ldots(k\varepsilon - x)$.*

(i) *If $\varepsilon < 0$ then ξ_ε^f does not have roots in $(0, +\infty)$.*

(ii) *If $\varepsilon > 0$ then ξ_ε^f has exactly k roots in $(0, +\infty)$, these roots are $\{\varepsilon, 2\varepsilon \ldots, k\varepsilon\}$ and $\dfrac{\partial \xi_\varepsilon^f}{\partial x}(j\varepsilon) = (-1)^j \varepsilon^k h(j\varepsilon)(k-j)!(j-1)!$ for $j \in \{1, 2, \ldots, k\}$. It means that the derivative at the root $j\varepsilon$ is positive for j even and negative for j odd.*

Proof. When $x > 0$, by a straightforward calculation $\xi_\varepsilon^f(x) = 0$ if, and only if, $(\varepsilon - x)(2\varepsilon - x)\ldots(k\varepsilon - x) = 0$. So, the roots of $\xi_\varepsilon^f(x)$ in $(0, +\infty)$ are $\varepsilon, 2\varepsilon, \ldots, k\varepsilon$. Moreover,

$$\frac{\partial \xi_\varepsilon^f}{\partial x}(x) = \frac{\partial}{\partial x}\Big((j\varepsilon - x)H(x)\Big) = (j\varepsilon - x)\frac{\partial H}{\partial x}(x) - H(x),$$

where $H(x)=\xi_\varepsilon^f(x)/(j\varepsilon-x)$. So,

$$\begin{aligned}\frac{\partial \xi_\varepsilon^f}{\partial x}(j\varepsilon) &= -H(j\varepsilon)=\varepsilon^k h(j\varepsilon)(1-j)\dots((j-1)-j)((j+1)-j)\dots(k-j)\\ &= \varepsilon^k h(j\varepsilon)(-1)^j\Big((j-1)\dots(j-(j-1))\Big)\Big(((j+1)-j)\dots(k-j)\Big)\\ &= (-1)^j\varepsilon^k h(j\varepsilon)(k-j)!(j-1)!\end{aligned}$$

This proves item (ii). Item (i) follows immediately. □

Lemma 6. *Consider the function $\xi_\varepsilon^i(x)=h(x)\sin(\pi\varepsilon^2/x)$. For $\varepsilon\neq 0$ the function ξ_ε^i has infinity many roots in $(0,\varepsilon^2)$, these roots are $\{\varepsilon^2,\varepsilon^2/2,\varepsilon^2/3,\dots\}$ and*

$$\frac{\partial \xi_\varepsilon^i}{\partial x}(\varepsilon^2/j)=(-1)^j(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j)\ \textit{for}\ j\in\{1,2,3,\dots\}.$$

It means that the derivative at the root ε^2/j is positive for j odd and negative for j even.

Proof. When $x>0$, by a straightforward calculation $\xi_\varepsilon^i(x)=0$ if, and only if, $\sin(\pi\varepsilon^2/x)=0$. So, the roots of $\xi_\varepsilon^i(x)$ in $(0,\varepsilon^2)$ are $\varepsilon^2,\varepsilon^2/2,\varepsilon^2/3,\dots$. Moreover,

$$\frac{\partial \xi_\varepsilon^i}{\partial x}(x)=h'(x)\sin(\pi\varepsilon^2/x)+h(x)\cos(\pi\varepsilon^2/x)(-\pi\varepsilon^2/x^2).$$

So,

$$\begin{aligned}\frac{\partial \xi_\varepsilon^i}{\partial x}(\varepsilon^2/j) &= h'(\varepsilon^2/j)\sin(\pi\varepsilon^2/j)+h(\varepsilon^2/j)\cos(\pi\varepsilon^2/j)(-\pi j^2/\varepsilon^2)\\ &= (-1)^j(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j).\end{aligned}$$

□

Since h is a C^∞-function, the functions $\xi_\varepsilon^f(x)$ in Lemma 5 and $\xi_\varepsilon^i(x)$ in Lemma 6 are C^∞-functions. So $Z_\varepsilon^\rho\in\Omega$ given by

$$(4)\qquad Z_\varepsilon^\rho(x,y)=\begin{cases} X(x,y)=\begin{pmatrix}-1\\ 2x\end{pmatrix} & \text{if } y\geq 0,\\ Y_\varepsilon^\rho(x,y)=\begin{pmatrix}1\\ 2x+\frac{\partial\xi_\varepsilon^\rho}{\partial x}(x)\end{pmatrix} & \text{if } y\leq 0,\end{cases}$$

where either $\rho=f$ or $\rho=i$, is a small C^∞-perturbation of Z_0 given by (1) when ε is sufficiently small. Moreover,

$$(5)\qquad \lim_{\varepsilon\to 0} Z_\varepsilon^\rho = Z_0.$$

Lemma 7. *Let $\Pi_{Z_\varepsilon^\rho}$ be the return map of Z_ε^ρ where either $\rho=f$ or $\rho=i$. For all $x>0$ we have that*

$$x^2-(\Pi_{Z_\varepsilon^\rho}(x))^2-\xi_\varepsilon^\rho(\Pi_{Z_\varepsilon^\rho}(x))=0.$$

Proof. Let $(x_0, 0) \in \Sigma$. According to Lemma 4, the trajectories of X are the graphs of $F_K(x) = -x^2 + K$ for $K \in \mathbb{R}$. The constant $K \in \mathbb{R}$ that satisfy $F_K(x_0) = 0$ is $K = x_0^2$. The parabola $y = -x^2 + x_0^2$ intersects the x-axis at the points x_0 and $-x_0$. So, $\Pi_X(x_0) = -x_0$. Again by Lemma 4, the trajectories of Y_ε^ρ are the graphs of $G_K(x) = x^2 + \xi_\varepsilon^\rho(x) + K$ for $K \in \mathbb{R}$. The constant $K \in \mathbb{R}$ that satisfy $G_K(-x_0) = 0$ is $K = -x_0^2$. So, the first return $\Pi_{Y_\varepsilon^\rho}(-x_0)$ is the first coordinate of the point in Σ given by the intersection of the graph of the function $G(x) = x^2 + \xi_\varepsilon^\rho(x) - x_0^2$ with the straight line $\{y = 0\}$. So $\Pi_{Z_\varepsilon^\rho}$ satisfies

$$x^2 - (\Pi_{Z_\varepsilon^\rho}(x))^2 - \xi_\varepsilon^\rho(\Pi_{Z_\varepsilon^\rho}(x)) = 0, \tag{6}$$

where either $\rho = f$ or $\rho = i$. □

Lemma 8. *Let $\Pi_{Z_\varepsilon^f}$ be the return map of Z_ε^f. Then $x > 0$ is a fixed point of $\Pi_{Z_\varepsilon^f}$ if, and only if, $x = j\varepsilon$ for $j = 1, 2, \ldots, k$. Moreover, for j even $(\Pi_{Z_\varepsilon^f})'(j\varepsilon) < 1$ and for j odd $(\Pi_{Z_\varepsilon^f})'(j\varepsilon) > 1$.*

Proof. According to Lemma 7, $x = \Pi_{Z_\varepsilon^f}(x)$ if, and only if, $\Pi_{Z_\varepsilon^f}(x)$ is a zero of the function $\xi_\varepsilon^f(x)$, i.e., by Lemma 5, $x = j\varepsilon$ for $j = 1, 2, \ldots, k$. Differentiating (6) with respect to x we obtain $2x - 2\Pi_{Z_\varepsilon^f}(x)(\Pi_{Z_\varepsilon^f})'(x) - \frac{\partial \xi_\varepsilon^f}{\partial x}(\Pi_{Z_\varepsilon^f}(x))(\Pi_{Z_\varepsilon^f})'(x) = 0$, and so

$$(\Pi_{Z_\varepsilon^f})'(j\varepsilon) = \frac{2j\varepsilon}{2j\varepsilon + \frac{\partial \xi_\varepsilon^f}{\partial x}(j\varepsilon)}.$$

According to Lemma 5, if j is even then $\frac{\partial \xi_\varepsilon^f}{\partial x}(j\varepsilon) > 0$ and it implies that $(\Pi_{Z_\varepsilon^f})'(j\varepsilon) < 1$. And if j is odd then $(\Pi_{Z_\varepsilon^f})'(j\varepsilon) > 1$. □

Remark 2. *A similar result of Lemma 8, for the PSVF Z_ε^i, also holds.*

With the previous lemmas we can stated the following proposition.

Proposition 9. *Consider Z_ε^ρ given by (4). Then, for $\varepsilon = 0$, $Z_\varepsilon^\rho = Z_0$ given by (1) has a nondegenerate Σ-center at the origin and*

- (I) *For $\rho = f$*
  - (I.i) *Z_ε^f has k limit cycles bifurcating from the nondegenerate Σ-center of Z_0 when $\varepsilon > 0$,*
  - (I.ii) *The limit cycle passing through $(j\varepsilon, 0)$ is attractor (respectively, repeller) if j is even (respectively, odd), with $j \in \{1, 2, \ldots k\}$.*
- (II) *For $\rho = i$*
  - (II.i) *Z_ε^i has infinitely many limit cycles bifurcating from the nondegenerate Σ-center of Z_0 when $\varepsilon \neq 0$,*
  - (II.ii) *The limit cycle passing through $(\varepsilon^2/j, 0)$ is attractor (respectively, repeller) if j is odd (respectively, even).*

Proof. It is easy to see that Z_0^ρ can be expressed by (1) and it has a nondegenerate Σ-center at the origin. According to Lemma 7, $x = \Pi_{Z_\varepsilon^\rho}(x)$ if, and only if, $\Pi_{Z_\varepsilon^\rho}(x)$ is a zero of the function $\xi_\varepsilon^\rho(x)$.

Therefore when $\rho = f$, by Lemma 5, the fixed points of $\Pi_{Z_\varepsilon^f}$ are given by $x = j\varepsilon$ for $j = 1, 2, \dots, k$. Since an isolated fixed point of $\Pi_{Z_\varepsilon^f}$ corresponds to a hyperbolic limit cycle of Z_ε^f, items (I.i) and (I.ii) follow immediately from Lemma 5 (item (ii)), and Lemma 8.

On other hand when $\rho = i$, by Lemma 6, the fixed points of $\Pi_{Z_\varepsilon^i}$ are given by $x = \varepsilon^2/j$ for $j = 1, 2, 3, \dots$. Since an isolated fixed point of $\Pi_{Z_\varepsilon^i}$ corresponds to a hyperbolic limit cycle of Z_ε^i, items (II.i) and (II.ii) follow immediately from Lemma 6 and Remark 2. □

Finally, we can prove Theorem A.

Proof of Theorem A. Let $\mathcal{W} \subset \Omega$ be an arbitrary neighborhood of Z_0. According to (5), for $\varepsilon > 0$ sufficiently small we have that $Z_\varepsilon^\rho \in \mathcal{W}$. The conclusion of the proof follows from Proposition 9 just taking $\widetilde{Z} = Z_\varepsilon^\rho$. □

5. Proof of Theorem B

Let Z_0 be given by (1). We will prove that for any neighborhood $\mathcal{W}$ of Z_0 and for any $k \in \mathbb{Z}$, $k > 0$, there exists $Z_\varepsilon^k \in \mathcal{W}$ of codimension k. We start this discussion with the following lemma.

Lemma 10. *For each $\varepsilon \in \mathbb{R}$ and $k \in \mathbb{Z}$, $k > 0$, consider $Z_\varepsilon^k = (X_0, Y_\varepsilon^k)$ where $X_0(x,y) = \begin{pmatrix} -1 \\ 2x \end{pmatrix}$ and $Y_\varepsilon^k(x,y) = \begin{pmatrix} 1 \\ 2x + \frac{\partial \xi_\varepsilon^k}{\partial x}(x) \end{pmatrix}$, with $\xi_\varepsilon^k(x) = \varepsilon h(x)(x-\varepsilon)^{k+1}$. The return map $\Pi_{Z_\varepsilon^k}$ associated to Z_ε^k is expressed by*

$$\Pi_{Z_\varepsilon^k}(x) = x - \frac{h(\varepsilon)}{2}(x-\varepsilon)^{k+1} + \mathcal{O}((x-\varepsilon)^{k+2}).$$

Proof. From (3) we know that $\Pi_{Z_\varepsilon^k}(x) = \Pi_{Y_\varepsilon^k}(\Pi_{X_0}(x))$. Since $\Pi_{X_0}(x) = -x$, in order to prove this lemma it is enough to find the expression of $\Pi_{Y_\varepsilon^k}$. Using Lemma 4, the trajectory of Y_ε^k are vertical translations of the graph of $F(z,\varepsilon) = z^2 + \xi_\varepsilon^k(z)$. Moreover, the trajectory of Y_ε^k through $(-x, 0)$, $x > 0$, is $(t, t^2 + \xi_\varepsilon^k(t) - (-x)^2 - \xi_\varepsilon^k(-x)) = (t, t^2 + \xi_\varepsilon^k(t) - x^2)$, since $\xi_\varepsilon^k(z) = 0$ for $z \leq 0$. So, $\Pi_{Z_\varepsilon^k}(x)$ is given implicitly as one of the solutions z of

$$G(z,\varepsilon) = 0, \tag{7}$$

where $G(z,\varepsilon) = z^2 + \xi_\varepsilon^k(z) - x^2$. It is easy to see that if $x = \varepsilon$ then $z = \varepsilon$ is a solution of (7), i.e., $\Pi_{Z_\varepsilon^k}(\varepsilon) = \varepsilon$. Moreover, a straightforward calculus shows that

$$\xi_\varepsilon^k(\varepsilon) = \frac{\partial}{\partial x}\xi_\varepsilon^k(\varepsilon) = \cdots = \frac{\partial^k}{\partial x^k}\xi_\varepsilon^k(\varepsilon) = 0 \text{ and } \frac{\partial^{k+1}}{\partial x^{k+1}}\xi_\varepsilon^k(\varepsilon) = \varepsilon h(\varepsilon)(k+1)!.$$

Using the Taylor's series of $\xi_\varepsilon^k(z)$ we get

$$G(z,\varepsilon) = z^2 + \varepsilon h(\varepsilon)(z-\varepsilon)^{k+1} - x^2 + \mathcal{O}((z-\varepsilon)^{k+2}).$$

So, $\dfrac{\partial}{\partial z}G(z,\varepsilon) = 2z + \varepsilon h(\varepsilon)k(z-\varepsilon)^k + \mathcal{O}((z-\varepsilon)^{k+1}) > 0$ for all z close to ε, which implies that $G(z,\varepsilon)$ is strictly increasing and $\Pi_{Z_\varepsilon^k}(x)$ is the unique positive root of (7) when $x>0$. The equation (7) can be writen as

$$z^2 + \varepsilon h(\varepsilon)(z-\varepsilon)^{k+1} + \mathcal{O}((z-\varepsilon)^{k+2}) = x^2 - \varepsilon^2. \tag{8}$$

If we call $\widetilde{z} = z-\varepsilon$ and $\widetilde{x} = x-\varepsilon$ then (8) becomes

$$2\varepsilon\widetilde{z} + \widetilde{z}^2 + \varepsilon h(\varepsilon)\widetilde{z}^{k+1} + \mathcal{O}(\widetilde{z}^{k+2}) = 2\varepsilon\widetilde{x} + \widetilde{x}^2. \tag{9}$$

Now, imposing $\widetilde{z} = a_1\widetilde{x} + a_2\widetilde{x}^2 + \cdots + a_{k+1}\widetilde{x}^{k+1} + \mathcal{O}(\widetilde{x}^{k+2})$, we get $a_1 = 1$, $a_2 = \cdots = a_k = 0$ and $a_{k+1} = -\frac{h(\varepsilon)}{2}$. So, we obtain

$$\Pi_{Z_\varepsilon^k}(x) = x - \frac{h(\varepsilon)}{2}(x-\varepsilon)^{k+1} + \mathcal{O}((x-\varepsilon)^{k+2}).$$

□

Observe that, similarly to the previous section, since h is a C^∞-function, the function $\xi_\varepsilon^k(x)$ in Lemma 10 is a C^∞-function. So $Z_\varepsilon^k \in \Omega$ is a small C^∞-perturbation of Z_0 given by (1) when ε is sufficiently small. Moreover,

$$\lim_{\varepsilon\to 0} Z_\varepsilon^k = Z_0. \tag{10}$$

Proposition 11. *The codimension of $\Pi_{Z_\varepsilon^k}$, $\varepsilon > 0$, is k and a (mini) versal unfolding for it is*

$$\Pi^\lambda_{Z_\varepsilon^k}(x) = \lambda_1 + \lambda_2 x + \cdots + \lambda_k x^{k-1} + x - x^{k+1}.$$

An important result that will be used in the proof of Proposition 11 is due to Malgrange and can be found in [6, 13].

Theorem 12 (Malgrange)**.** *Let $F(\mu,x)$, $\mu\in\mathbb{R}^m$, be a smooth, real-valued function defined on a neighborhood of the origin in $\mathbb{R}^m\times\mathbb{R}$. Furthemore, suppose that $F(0,x) = x^{k+1}g(x)$, where $g(x)$ is smooth and $g(0)\neq 0$. Then there exists a function $q(\mu,x)$, smooth, and smooth functions $s_i(\mu)$, $i = 0,\ldots,k+1$ such that*

$$q(\mu,x)F(\mu,x) = x^{k+1} + \sum_{i=1}^{k+1} s_i(\mu)x^{i-1}.$$

Another result that will be useful to prove Proposition 11, whose proof can be found in the Example 1.5.1 on page 20 of [1], is the following.

Lemma 13. *Let $f:\mathbb{R}\to\mathbb{R}$ be a local diffeomorphism, with $Df(x)>0$ for some $x\in\mathbb{R}$, and consider $\varphi_t:\mathbb{R}\to\mathbb{R}$ the flow defined by the differential equation $\dot{x} = f(x) - x$. Then f is topologically conjugate to φ_1.*

For a precise definition of topological conjugacy of maps see Definition 19, in the Appendix.

Next lemma deals with the concepts of equivalence and conjugacy between smooth vector fields. The concept of equivalence is similar to Definition 3. The homeomorphism must send each trajectories of one vector field in a trajectories of the second one preserving the orientation. The concept of conjugacy can be found in Definition 20 in the Appendix.

Lemma 14. *If $\dot{x} = f(x)$, $x \in \mathbb{R}$, has a finite number of singularities and it is topologically equivalent to $\dot{x} = g(x)$ then they are topologically conjugate.*

Proof. Let $p_1, \ldots, p_n$ be the singularities of $\dot{x} = f(x)$. Assume that $p_1 < \cdots < p_n$. Let h be the homeomorphism that gives the equivalence between $\dot{x} = f(x)$ and $\dot{x} = g(x)$. So, the singularities of $\dot{x} = g(x)$ are $q_i = h(p_i)$, $i = 1, \ldots, n$. The homeomorphism is either increasing or decreasing. Without loss of generality assume that h is increasing. So, we have $q_1 < \cdots < q_n$. If we consider the restriction $h_i = h|_{[p_i, p_{i+1}]}$ then $h_i : [p_i, p_{i+1}] \to [q_i, q_{i+1}]$ is an homeomorphism. Now we will build another homeomorphism $\widetilde{h}_i : [p_i, p_{i+1}] \to [q_i, q_{i+1}]$ that is a topological conjugacy between $\dot{x} = f(x)$ restricted to $[p_i, p_{i+1}]$ and $\dot{x} = g(x)$ restricted to $[q_i, q_{i+1}]$. Pick points $x_1 \in (p_i, p_{i+1})$ and $y_1 \in (q_i, q_{i+1})$. We show that there is a unique conjugacy $\widetilde{h}_i(x)$ with $\widetilde{h}_i(x_1) = y_1$. For any x there is a unique time $-\infty < t(x) < \infty$ such that $\phi(t(x), x_1) = x$, where $\phi_t = \phi(t, \cdot)$ is the flow of $\dot{x} = f(x)$. Observe that if $f(x) > 0$ in $[p_i, p_{i+1}]$ then the function $t(x)$ is a strictly increasing function of x. The fact that $t(x)$ is differentiable follows from the differentiability of ϕ with $\phi_x > 0$, together with the Implicit Function Theorem. If there were a conjugacy with $\widetilde{h}_i(x_1) = y_1$ then it would satisfy

$$\widetilde{h}_i(x) = \widetilde{h}_i(\phi(t(x), x_1)) = \psi(t(x), \widetilde{h}_i(x_1)) = \psi(t(x), y_1), \tag{11}$$

where $\psi_t = \psi(t, \cdot)$ is the flow of $\dot{x} = g(x)$. This shows that there is only one possibility. Furthermore, since $t(x) \to \infty$ as $x \to p_{i+1}$ it follows that $\widetilde{h}_i$ has a continuous extension to $x = p_{i+1}$ by setting $\widetilde{h}_i(p_{i+1}) = q_{i+1}$. Similarly defining $\widetilde{h}_i(p_i) = q_i$ yields a continuous strictly increasing map of $[p_i, p_{i+1}]$ onto $[q_i, q_{i+1}]$. The inverse of a strictly increasing continuous function is also a continuous strictly increasing function proving the invertibility of $\widetilde{h}_i$. It remains to show that

$$\widetilde{h}_i(\phi(t, x)) = \psi(t, \widetilde{h}_i(x)). \tag{12}$$

If x is an endpoint this is immediate. Equation (11) implies that for all $-\infty < t < \infty$

$$\widetilde{h}_i(\phi(t, x_1)) = \psi(t, \widetilde{h}_i(x_1)). \tag{13}$$

If x is not an endpoint, write $x = \phi(t(x), x_1)$ so $\phi(t, x) = \phi(t + t(x), x_1)$. Compute using (11)$-$(13), $\widetilde{h}_i(\phi(t, x)) = \widetilde{h}_i(\phi(t+t(x), x_1)) = \psi(t+t(x), y_1) =$

$\psi(t,\psi(t(x),y_1)) = \psi(t,\widetilde{h}_i(x))$. Finally we glue conjugacies on adjacent intervals to yield a conjugacy $\widetilde{h}$ on the union. □

Proof of Proposition 11. Consider $f(\mu,x)$, $\mu\in\mathbb{R}^m$, an arbitrary unfolding of $\Pi_{Z_\varepsilon^k}$. By Lemma 13, $f(\mu,x)$ is topologically conjugate to φ_1, where φ_t is the flow of $\dot{x}=F(\mu,x)$ with $F(\mu,x)=f(\mu,x)-x$. From Lemma 10 we observe that $F(\mu,x)$ satisfies the hypotheses of Theorem 12. So

$$F(\mu,x)=\frac{1}{q(\mu,x)}\left(-x^{k+1}+\sum_{i=1}^{k+1}s_i(\mu)x^{i-1}\right)$$

and $\dot{x}=F(\mu,x)$ is topologically equivalent (by the identity) to $\dot{x}=\overline{G}(\mu,x)$, where

$$\overline{G}(\mu,x)=-x^{k+1}+\sum_{i=1}^{k+1}s_i(\mu)x^{i-1}.$$

Observe that $\overline{G}(\mu,x)$ is induced from $\overline{H}(\lambda,x)=-x^{k+1}+\sum_{i=1}^{k+1}\lambda_i x^{i-1}$, $\lambda=(\lambda_1,\dots,\lambda_{k+1})\in\mathbb{R}^{k+1}$, just considering $\lambda_i=s_i(\mu)$, $i=1,\dots,k+1$. In fact, a (mini) versal unfolding of $\dot{x}=-x^{k+1}$ is $\dot{x}=H(\lambda,x)$, where $H(\lambda,x)=-x^{k+1}+\sum_{i=1}^{k}\lambda_i x^{i-1}$, $\lambda\in\mathbb{R}^k$ (see [9]). It means that there exists $G(\mu,x)$ such that it is topologically equivalent to $F(\mu,x)$ and it is induced from $H(\lambda,x)$. Let $\widetilde{\varphi}_t:\mathbb{R}\to\mathbb{R}$ be the flow of $\dot{x}=G(\mu,x)$. Observe that if $\dot{x}=F(\mu,x)$ is topologically equivalent to $\dot{x}=G(\mu,x)$ then, by Lemma 14, $\dot{x}=F(\mu,x)$ is topologically conjugate to $\dot{x}=G(\mu,x)$. And consequently φ_1 is topologically conjugate to $\widetilde{\varphi}_1$. Again by Lemma 13, $\widetilde{\varphi}_1$ is topologically conjugate to $g(\mu,x)=G(\mu,x)+x$. The unfolding $g(\mu,x)$ is induced from $h(\lambda,x)=H(\lambda,x)+x=x-x^{k+1}+\sum_{i=1}^{k}\lambda_i x^{i-1}$. So, we conclude that $h(\lambda,x)$ is versal, because any unfolding $f(\mu,x)$ of $\Pi_{Z_\varepsilon^k}$ is topologically conjugate to an unfolding induced from $h(\lambda,x)$. The minimality of the number of parameters follows from the fact that $\dot{x}=H(\lambda,x)$ is a (mini) versal unfolding of $\dot{x}=-x^{k+1}$. □

Theorem 15. *For each $\varepsilon\in\mathbb{R}$, $\varepsilon>0$, and $k\in\mathbb{Z}$, $k>0$, the PSVF $Z_\varepsilon^k=(X_0,Y_\varepsilon^k)$ given in Lemma 10 has codimension k.*

Proof. According to Proposition 22 and its corollary (see Appendix), the codimension of Z_ε^k is equal to the codimension of $\Pi_{Z_\varepsilon^k}$. From Proposition 11 the codimension of $\Pi_{Z_\varepsilon^k}$ is k. □

Proof of Theorem B. Consider Z_0 given by (1). Let $\mathcal{W}\subset\Omega$ be an arbitrary neighborhood of Z_0 and $k\in\mathbb{Z}$, $k>0$. According to (10), for $\varepsilon>0$ sufficiently small we have that $Z_\varepsilon^k\in\mathcal{W}$, where Z_ε^k is given by Theorem 15 and it has codimension k. □

Appendix: Topologically Versal Unfoldings

A.1. **Topological equivalence of discontinuous vector fields.** Consider a family of differential equations depending smoothly on parameters ε:

$$Z_\varepsilon(x,y) = \begin{cases} X_1(x,y,\varepsilon), & \text{if } y \geq 0; \\ Y_1(x,y,\varepsilon), & \text{if } y \leq 0; \end{cases} \tag{14}$$

Definition 16. *A family of differential equations depending smoothly on parameters ε*

$$W_\varepsilon(x,y) = \begin{cases} X_2(x,y,\varepsilon), & \text{if } y \geq 0; \\ Y_2(x,y,\varepsilon), & \text{if } y \leq 0; \end{cases} \tag{15}$$

is said to be (topologically) equivalent to the family (14) *if there exists a homeomorphism h_ε, depending continuously on ε, that takes the (oriented) phase curves of the system* (14) *into those of* (15)*.*

In other words, a topological equivalence of families is a Σ-equivalence of systems, depending continuously on the parameter.

Definition 17. *A family of differential equations depending smoothly on parameters μ*

$$U_\mu(x,y) = \begin{cases} X_3(x,y,\mu), & \text{if } y \geq 0; \\ Y_3(x,y,\mu), & \text{if } y \leq 0; \end{cases} \tag{16}$$

is said to be induced from (14) *if it is obtained from* (14) *by a continuous change of parameters, that is, if there exists a continuous map $\varepsilon = \phi(\mu)$ such that*

$$U_\mu(x,y) = Z_{\phi(\mu)}(x,y).$$

Definition 18. *The germ of the family* (14) *at $\varepsilon = 0$ is called an unfolding of the PSVF $Z_0(x,y)$. The unfolding* (14) *is said to be a topologically versal unfolding of the PSVF $Z_0(x,y)$ if every other unfolding of the same PSVF is equivalent to that one induced from* (14)*. A versal unfolding involving the smallest number of parameters is called (mini) versal unfolding.*

Let (x_0,y_0) be a singularity of $Z_0(x,y)$. In this case the minimal dimension that a topologically versal family can have is the minimal number of parameters of families in which singularities of the given topological type occur so that they cannot be avoided by small shifts. This minimal number is called the *codimension of the singularity*.

A.2. **Topological conjugacy of maps.**

Definition 19. *Let f_1 and f_2 be two maps defined in neighborhoods V_1 and V_2 respectively. We say that a topological conjugacy of the maps f_1 and f_2 is a homeomorphism $g : V_1 \to V_2$ that satisfies*

$$g \circ f_1 = f_2 \circ g.$$

By considering a family of maps depending smoothly on parameters ε, like in Section A.1, we can give definitions of topological conjugacy of families of maps, induced families of maps and (mini) versal unfoldings of maps.

A.3. **Topological conjugacy of smooth vector fields.**

Definition 20. *Let $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$ be two smooth vector fields defined in neighborhoods V_1 and V_2 respectively. We say that a topological conjugacy of the vector fields f_1 and f_2 is a homeomorphism $g : V_1 \to V_2$ that satisfies*

$$g \circ \varphi_t^1 = \varphi_t^2 \circ g,$$

where φ_t^1 and φ_t^2 are the flows of f_1 and f_2 respectively.

A.4. **Relationship of discontinuous vector fields and first return maps.**

Proposition 21. *The PSVFs $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$ are Σ-equivalent if, and only if, the corresponding return maps Π_{Z_1} and Π_{Z_2} are topologically conjugate.*

Proof. First we prove that if h is a topological equivalence of the PSVFs Z_1 and Z_2 then $H|_\Sigma$ is a topological conjugacy of the return maps Π_{Z_1} and Π_{Z_2}. Assume that H is a homeomorphism that sends trajectories of Z_1 to trajectories of Z_2, preserving the direction of time and preserving Σ. Consider a point $\rho \in \Sigma$ in a region where the return map is well defined. Due to the fact that H is Σ-preserving we have that the point $H(\rho)$ belongs to Σ. Let Γ_{X_1} (resp. Γ_{X_2}) be the orbit of X_1 (resp. X_2) passing through ρ (resp. $H(\rho)$). We also have that $\Gamma_{X_2} = H(\Gamma_{X_1})$. The point where the orbit Γ_{X_1} (resp. Γ_{X_2}) hits Σ is the positive half return map $\Pi_{X_1}(\rho)$ (resp. $\Pi_{X_2}(h(\rho))$). By using the fact that $\Gamma_{X_2} = H(\Gamma_{X_1})$ and H is Σ-preserving we obtain $\Pi_{X_2}(H(\rho)) = H(\Pi_{X_1}(\rho))$. In another words we have

$$\Pi_{X_2} \circ H = H \circ \Pi_{X_1}.$$

By the same argument applied to the vector fields Y_1 and Y_2 we obtain $\Pi_{Y_2} \circ H = H \circ \Pi_{Y_1}$. Now we have

$$\Pi_{Z_2} \circ H = \Pi_{Y_2} \circ \Pi_{X_2} \circ H = \Pi_{Y_2} \circ H \circ \Pi_{X_1} = H \circ \Pi_{Y_1} \circ \Pi_{X_1} = H \circ \Pi_{Z_1}.$$

On the other hand, if there exist neighborhoods V_1 and V_2 contained in Σ and a homeomorphism $g : V_1 \to V_2$ that conjugates Π_{Z_1} and Π_{Z_2}, then we can extend g to an homeomorphism H defined in an open set of $\mathbb{R}^2$. The extension is made by using the orbits of Z_1 and Z_2 and the arc length in an analogous way that we done in the proof of Theorem 1. So H gives a Σ-equivalence between Z_1 and Z_2. □

Proposition 22. *The family*

$$Z_\varepsilon(x, y) = \begin{cases} X(x, y, \varepsilon), & \text{if } y \geq 0; \\ Y(x, y, \varepsilon), & \text{if } y \leq 0; \end{cases} \tag{17}$$

is a versal unfolding of $Z_0(x, y)$ if, and only if, Π_{Z_ε} is a versal unfolding of Π_{Z_0}.

Proof. Assume that Z_ε is a versal unfolding of Z_0. For each ε consider the return map Π_{Z_ε} associated to Z_ε. So Π_{Z_ε} is an unfolding of Π_{Z_0}. In order to prove that Π_{Z_ε} is a versal unfolding of Π_{Z_0}, consider Π_μ an arbitrary unfolding of Π_{Z_0}. Easily we can build a family of PSVFs Z_μ such that $\Pi_{Z_\mu} = \Pi_\mu$. The versality of Z_ε implies that Z_μ is isomorphic to an induced unfolding of Z_ε. Applying previous proposition we have that $\Pi_{Z_\mu} = \Pi_\mu$ is isomorphic to an induced unfolding of Π_{Z_ε}. So, Π_{Z_ε} is a versal unfolding of Π_{Z_0}.

For the converse, assume that Π_{Z_ε} is a versal unfolding of Π_{Z_0}. For each ε consider the PSVF Z_ε. So Z_ε is a unfolding of Z_0. In order to prove that Z_ε is a versal unfolding of Z_0, consider Z_μ an arbitrary unfolding of Z_0. So Π_{Z_μ} is an unfolding of Π_{Z_0}. The versality of Π_{Z_ε} implies that Π_{Z_μ} is isomorphic to an induced unfolding of Π_{Z_ε}. Applying previous proposition we have that Z_μ is isomorphic to an induced unfolding of Z_ε. So, Z_ε is a versal unfolding of Z_0. □

Corollary 23. *The codimension of the PSVF Z_0 is equal to the codimension of the return map Π_{Z_0}. In other words, Z_ε is a (mini) versal unfolding of Z_0 if, and only if, Π_{Z_ε} is a (mini) versal unfolding of Π_{Z_0}.*

Proof. Let m be the codimension of Z_0. So, there exists a versal unfolding Z_ε of Z_0, with $\varepsilon \in \mathbb{R}^m$, and there is no versal unfolding Z_μ of Z_0, with $\mu \in \mathbb{R}^n$ and $n < m$. By previous proposition Π_{Z_ε} is a versal unfolding of Π_{Z_0} with m parameters. So, the codimension of Π_{Z_0} is less or equal to m. Suppose that there exists a versal unfolding Π_μ of Π_{Z_0} with n parameters, where $n < m$. In this case we can build a versal unfolding Z_μ of Z_0 with n parameters. This contradiction completes the proof. □

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[1] IBILCE–UNESP, CEP 15054–000, S. J. Rio Preto, São Paulo, Brazil

[2] FC–UNESP, CEP 17033–360, Bauru, São Paulo, Brazil

[3] IMECC–UNICAMP, CEP 13081–970, Campinas, São Paulo, Brazil
E-mail address: buzzi@ibilce.unesp.br
E-mail address: tcarvalho@fc.unesp.br
E-mail address: teixeira@ime.unicamp.br