RUELLE OPERATOR DUALITY
FOR COUPLED SMOOTH MARKOV MAPS OF THE CIRCLE

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ABSTRACT. Let T_L and T_R be two smooth surjective Markov maps of the circle, with T_R expansive, coupled in such a way that there exists an extension (C, T_C) whose first factor is T_L and the second factor of T_C^{-1} is T_R. Let A_L piecewise continuous and A_R piecewise absolutely continuous be two respective potentials. We show that, when those potentials are in involution by a smooth kernel W on C, there is an explicit isomorphism between eigenfunctions of the Ruelle operator of (T_L, A_L) and eigendistributions of the Ruelle operator of (T_R, A_R) for the same eigenvalue. This gives a regularity result for eigendistributions of transfer operators associated with non-maximal eigenvalues.

1. INTRODUCTION

If I is an half-open or a closed interval of the unit circle $\mathbb{S}^1$, we will say that a map f defined over I is continuous (respectively continuously differentiable) when there is an open interval U that contains I and a continuous (respectively continuously differentiable) map g defined over U such that $g_{/I} = f$; or equivalently when f is continuous (respectively continuously differentiable) over the interior of I and for every endpoint a of I, $\lim\limits_{x\to a} f(x)$ (respectively $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} f'(x)$) exists.

Let $T_L : \mathbb{S}^1 \to \mathbb{S}^1$ and $T_R : \mathbb{S}^1 \to \mathbb{S}^1$ be two surjective Markov maps defined of $\mathbb{S}^1$, respectively preserving the finite partitions $\mathcal{P}^L = (I_k^L)_{1\le k\le N}$ and $\mathcal{P}^R(I_k^R)_{1\le k\le N}$ of the circle in half-open intervals. Without loss of generality, we will assume that intervals of $\mathcal{P}_L$ are half-open to the left and intervals of $\mathcal{P}_R$ are half-open to the right. We suppose that each branch T_{L/I_k^L} induces an orientation-preserving $\mathcal{C}^1$-diffeomorphism from $\overline{I_k^L}$ to $\overline{T_L(I_k^L)}$. Let $\mathcal{P}_n^L$ (respectively $\mathcal{P}_n^R$) the set of T_L-cylinders (respectively T_R-cylinders) of length n relatively to the partition $\mathcal{P}^L$ (respectively $\mathcal{P}^R$). Since inverse branches of any length are homeomorphisms, the cylinders of $\mathcal{P}_n^L$ (respectively $\mathcal{P}_n^R$) form a partition of the circle in half-open intervals to the left (respectively to the right).

Two such maps are said to be *coupled* when there exist a subset C of the two-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and a bijection $T_C : C \to C$ such that :

(1) C can be written as a disjoint union of rectangles, i.e. there are two collections of intervals of the circle $(J_k^L)_{1\le k\le N}$ and $(J_k^R)_{1\le k\le N}$ such that :
$$C = \bigsqcup_{1\le k\le N} I_k^L \times J_k^R = \bigsqcup_{1\le k\le N} J_k^L \times I_k^R$$

(2) there are two maps $S_L : C \to \mathbb{S}^1$ and $S_R : C \to \mathbb{S}^1$ such that :
$$T_C(x,y) = (T_L(x), S_R(x,y)) \qquad \text{and} \qquad T_C^{-1}(x,y) = (S_L(x,y), T_R(y))$$
and :
– for every $x \in I_k^L$, $y \in J_k^R \mapsto S_R(x,y)$ induces an orientation-preserving $\mathcal{C}^1$-diffeomorphism from $\overline{J_k^R}$ to $\overline{S_R(x, J_k^R)}$;

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– for every $y \in I_k^R$, $x \in \overline{J_k^L} \mapsto \overline{S_L(x, y)}$ induces an orientation-preserving $\mathcal{C}^1$-diffeomorphism from $\overline{J_k^L}$ to $\overline{S_L(J_k^L, y)}$.

We will commonly write :

$$J_k^R = J^R(x) = [c(x); d(x)[\text{ when } x \in I_k^L$$
$$J_k^L = J^L(y) =]a(y); b(y)] \text{ when } y \in I_k^R$$

When T_L and T_R are coupled, we can define for every $n \geq 0$ the maps $S_L^n = \pi_1 \circ T_C^{-n} : C \to \mathbb{S}^1$ and $S_R^n = \pi_2 \circ T_C^n : C \to \mathbb{S}^1$, where π_1 (respectively π_2) is the projection map on the first (respectively second) coordinate of $\mathbb{T}^2$. Those two partial applications satisfy the following relations for every $(x, y) \in C$ and $n \geq k$:

$$T_L^k(S_L^n(x, y)) = S_L^{n-k}(x, y) \qquad \text{and} \qquad T_R^k(S_R^n(x, y)) = S_R^{n-k}(x, y)$$

We now consider a pair of potentials $A_L : \mathbb{S}^1 \to \mathbb{C}$ and $A_R : \mathbb{S}^1 \to \mathbb{C}$, respectively associated with T_L and T_R. We call an *involution kernel* a map $W : C \to \mathbb{C}$ such that :

$$\forall (x, y) \in C, W(x, y) + A_L(x) = W(T_C(x, y)) + A_R(S_R(x, y))$$

The n-th Birkhoff sums ($n \geq 0$) of these potentials will be denoted by :

$$A_L^n(x) = \sum_{k=0}^{n-1} A_L(T_L^k(x)) \qquad \text{and} \qquad A_R^n(x) = \sum_{k=0}^{n-1} A_R(T_R^k(x))$$

and they satisfy :

$$\forall n \geq 0, \forall (x, y) \in C, W(x, y) + A_L^n(x) = W(T_C^n(x, y)) + A_R^n(S_R^n(x, y))$$

In all the following, we will suppose that A_L is continuous on the closure of every interval of $\mathcal{P}_L$, and that A_R is absolutely continuous on the closure of every interval of $\mathcal{P}_R$. With these assumptions, the Ruelle operators associated with the pairs of map-potential (T_L, A_L) and (T_R, A_R) preserve and are respectively defined on the spaces :

$$E_L = \left\{\psi : \mathbb{S}^1 \to \mathbb{C} \;\middle|\; \forall k, \psi \text{ is continuous on the closure of } I_k^L\right\}$$
$$E_R = \left\{\varphi : \mathbb{S}^1 \to \mathbb{C} \;\middle|\; \forall k, \varphi \text{ is absolutely continuous on the closure of } I_k^R\right\}$$

and are given by :

$$\forall f \in E_L, \mathcal{L}_L\psi(x') = \sum_{T_L(x)=x'} e^{A_L(x)} f(x)$$
$$\forall f \in E_R, \mathcal{L}_R\psi(y) = \sum_{T_R(y')=y} e^{A_R(y')} f(y')$$

The Ruelle operator $\mathcal{L}_R$ of (T_R, A_R) also acts by duality on the space :

$$F_R = \left\{ \nu : E_R \to \mathbb{C} \;\middle|\; \begin{array}{c} \exists h \in \mathcal{C}^0(\mathbb{R}, \mathbb{C}), \forall \varphi \in E_R, \forall k, \\ \langle \nu, \varphi \mathbb{1}_{I_k^R} \rangle = \tilde{\varphi}(b_k^R) h(b_k^R) - \tilde{\varphi}(a_k^R) h(a_k^R) - \int_{a_k^R}^{b_k^R} \tilde{\varphi}'(t) h(t) dt \end{array} \right\}$$

Then, if $\lambda \in \mathbb{C} \setminus \{0\}$, we have the eigenspaces :

$$E_L(\lambda) = \{\psi \in E_L \mid \mathcal{L}_L\psi = \lambda\psi\}$$

and dual eigenspaces :

$$F_R(\lambda) = \{\nu \in F_R \mid \forall \varphi \in E_R, \langle \nu, \mathcal{L}_R\varphi \rangle = \lambda \langle \nu, \varphi \rangle\}$$

We then consider the map :

$$\begin{array}{rccl} \Phi : & F_R & \to & E_L \\ & \nu & \mapsto & \left(x \in I_k^L \mapsto \langle \nu, e^W(x, .) \mathbb{1}_{J_k^R}(.) \rangle\right) \end{array}$$

Because W is an involution kernel, it is easy to see that this map actually sends $F_R(\lambda)$ to $E_L(\lambda)$:

Lemma 1.1. $\Phi(F_R(\lambda)) \subset E_L(\lambda)$.

We will prove the following result :

Theorem 1.2. *Let T_L and T_R be two smooth surjective Markov maps of the circle, with T_R expansive, and coupled by a system (C, T_C). Let A_L continuous and A_R absolutely continuous be two potentials, respectively associated with T_L and T_R, and in involution by a kernel W such that $y \mapsto W(x, y)$ is absolutely continuous on $\overline{J^R(x)}$ for every $x \in \mathbb{S}^1$. Then :*

1. *The map Φ is injective.*
2. *If the map $x \mapsto W(x, d(x))$ is bounded on $\mathbb{S}^1$, and the function series :*
$$y \mapsto \sum_{n\geq 0} \lambda^{-n} e^{A_L^n(S_L^n(x,y))}$$
converges uniformly on $\overline{J^R(x)}$ for every x, then any $\psi \in E_L(\lambda)$ such that :
$$x \mapsto \psi(x)e^{-W(x,d(x)}$$
has bounded variations on the closure of every I_k^L admits a preimage in F_R by Φ.
Moreover, this preimage is explicitely given by the following limit :
$$h_n(x', y') = \sum_{T_L^n(x)=x'} \psi(x)e^{A_L^n(x)-W(x',S_R^n(x,d(x)))} \mathbb{1}_{]c(x');y']}(S_R^n(x, d(x)))$$
$$h(x', y') = \lim_{n\to+\infty} h_n(x', y')$$
$$h_0 = 0, h_{k+1} = h_k + h(x_k, y_k) - h(x_{k+1}, y_k)$$
where $I_k^R = [y_k; y_{k+1}[$ and x_k is such that $(x_k, y_k) \in C$
$$h(y) = h_k + h(x_k, y) \text{ if } y \in I_k^R$$
$$\nu = h'$$

One can have a similar result linking eigenfunctions of $\mathcal{L}_R$ and eigendistributions of $\mathcal{L}_L$ by switching the roles of (T_L, A_L) and (T_R, A_R), by changing T_C into T_C^{-1}, and by reversing the hypothesis on W.

2. Proof of the theorem

The first point to notice is that we can restrict ourselves to the case $\lambda = 1$. Indeed, $\psi \in E_L(\lambda)$ if and only if $\psi \in E'_L(1)$ where E'_L is the relevant space associated with the potential $A_L - \log\lambda$ (where $\log$ is any suitable complex determination of the logarithm). Likewise, $\nu \in F_R(\lambda)$ if and only if $\nu \in F'_R(1)$ where F'_R is associated with the potential $A_R - \log\lambda$. But W is an involution kernel between A_L and A_R if and only if it is an involution kernel between $A_L - \log\lambda$ and $A_R - \log\lambda$. Thus Φ is an isomorphism between $F_R(\lambda)$ and $E_L(\lambda)$ if and only if it is an isomorphism between $F'_R(1)$ and $E'_L(1)$. Hence we will assume from now on that λ is 1.

2.1. Partition of the fibers.

Lemma 2.1.
$$\forall n \geq 0, \forall x' \in \mathbb{S}^1, [c(x'); d(x')[= \bigsqcup_{T_L^n(x)=x'} [S_R^n(x, c(x)); S_R^n(x, d(x))[$$

Proof : Fix $n \geq 0$ and $x' \in \mathbb{S}^1$.

Let $y' \in [c(x'); d(x')[$. Since T_C is a bijection, there exists $(x, y) \in C$ such that :
$$T_C^n(x, y) = (x', y') = (T_L^n(x), S_R^n(x, y))$$
But $(x, y) \in C$ implies that $y \in [c(x); d(x)[$, thus $y' = S_R^n(x, y) \in [S_R^n(x, c(x)); S_R^n(x, d(x))[$.

Reciprocally, assume that $y' \in [S_R^n(x,c(x));S_R^n(x,d(x))[$ where $T_L^n(x)=x'$. Since $S_R^n(x,.)$ is an orientation-preserving homeomorphism from $[c(x);d(x)[$ onto $[S_R^n(x,c(x));S_R^n(x,d(x))[$, there exists an unique $y\in[c(x);d(x)[$ such that $S_R^n(x,y)=y'$. But then $(x,y)\in C$ and :

$$(x',y')=(T_L^n(x),S_R^n(x,y))=T_C^n(x,y)\in C$$

So y' must lie in $[c(x');d(x')[$.

Finally, suppose that $y' \in [S_R^n(x_1,c(x_1));S_R^n(x_1,d(x_1))[\cap [S_R^n(x_2,c(x_2));S_R^n(x_2,d(x_2))[$. As above, we can find an unique $y_1\in[c(x_1);d(x_1)[$ (respectively $y_2\in[c(x_2);d(x_2)[$) such that $y'=S_R^n(x_1,y_1)$ (respectively $y'=S_R^n(x_2,y_2)$). But :

$$T_C^n(x_1,y_1)=(T_L^n(x_1),S_R^n(x_1,y_1))=(x',y')=(T_L^n(x_2),S_R^n(x_2,y_2))=T_C^n(x_2,y_2)$$

so $x_1=x_2$ since T_C is a bijection over C. ■

Lemma 2.2. *For every* $(x,y)\in C$,

$$y\in[S_R^n(S_L^n(x,y),c(S_L^n(x,y)));S_R^n(S_L^n(x,y),d(S_L^n(x,y)))[$$

PROOF : Let $x_n=S_L^n(x,y)$. It is clear that $T_L^n(x_n)=\pi_1 T_C^n T_C^{-n}(x,y)=x$. On the other hand, $T_C^{-n}(x,y)=(S_L^n(x,y),T_R^n(y))\in C$ hence $T_R^n(y)\in[c(x_n);d(x_n)[$ and :

$$\begin{aligned} y=\pi_2 T_C^n T_C^{-n}(x,y) &= S_R^n(S_L^n(x,y),T_R^n(y)) \\ &= S_R^n(x_n,T_R^n(y))\in[S_R^n(x_n,c(x_n));S_R^n(x_n,d(x_n))[\,. \end{aligned}$$

Thus x_n is the only w such that $T_L^n(w)=x$ and $y\in[S_R^n(w,c(w));S_R^n(w,d(w))[$. ■

If $x\in\mathbb{S}^1$ is fixed, there exists for every $n\geq 0$ an unique $x_n\in\mathbb{S}^1$ such that $T_L^n(x_n)=x$ and $S_R^n(x_n,d(x_n))=d(x)$. We can then extend T_C^{-n} to $\tilde{C}$ by letting $S_L^n(x,d(x))=x_1$.

Lemma 2.3. *Let* $x'\in\mathbb{S}^1$, $n,k\geq 0$, *and* $x_n,x_{n+k}\in\mathbb{S}^1$ *such that* $T_L^n(x_n)=T_L^{n+k}(x_{n+k})=x'$.

$$\left[S_R^{n+k}(x_{n+k},c(x_{n+k}));S_R^{n+k}(x_{n+k},d(x_{n+k}))\right[\subset[S_R^n(x_n,c(x_n));S_R^n(x_n,d(x_n))[$$

if and only if $T_L^k(x_{n+k})=x_n$.

This result is true with open, half-open (on any side) or closed intervals.

PROOF : One has :

$$\begin{aligned} [S_R^n(x_n,c(x_n));S_R^n(x_n,d(x_n))[&= S_R^n(x_n,[c(x_n);d(x_n)[) \\ &= S_R^n\left(x_n,\bigsqcup_{T_L^k(x)=x_n}\left[S_R^k(x,c(x));S_R^k(x,d(x))\right[\right) \\ &= \bigsqcup_{T_L^k(x)=x_n}\left[S_R^n(T_L^k(x),S_R^k(x,c(x)));S_R^n(T_L^k(x),S_R^k(x,d(x)))\right[\\ &= \bigsqcup_{T_L^k(x)=x_n}\left[S_R^{n+k}(x,c(x));S_R^{n+k}(x,d(x))\right[\,. \end{aligned}$$

If $T_L^k(x_{n+k})=x_n$, it is clear that $\left[S_R^{n+k}(x_{n+k},c(x_{n+k}));S_R^{n+k}(x_{n+k},d(x_{n+k}))\right[$ is a subset of $[S_R^n(x_n,c(x_n));S_R^n(x_n,d(x_n))[$.

Reciprocally, if the inclusion is satisfied, there is a x such that $T_L^k(x)=x_n$ and :

$$\left[S_R^{n+k}(x_{n+k},c(x_{n+k}));S_R^{n+k}(x_{n+k},d(x_{n+k}))\right[=\left[S_R^{n+k}(x,c(x));S_R^{n+k}(x,d(x))\right[$$

Since those intervals are either identical or disjoints, one must have $x_{n+k}=x$, which proves that $T_L^k(x_{n+k})=x_n$. ■

Lemma 2.4. *Let $x' \in \mathbb{S}^1$, $n, k \geq 0$, and $x_n, x_{n+k} \in \mathbb{S}^1$ such that $T_L^n(x_n) = T_L^{n+k}(x_{n+k}) = x'$. Then :*

$$\left[S_R^{n+k}(x_{n+k}, c(x_{n+k})); S_R^{n+k}(x_{n+k}, d(x_{n+k}))\right[\cap [S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[\neq \emptyset$$

if and only if

$$\left[S_R^{n+k}(x_{n+k}, c(x_{n+k})); S_R^{n+k}(x_{n+k}, d(x_{n+k}))\right[\subset [S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[$$

This result is true with open or half-open intervals (on any side), but it is false with closed intervals.

PROOF : The reverse implication being obvious, we only have to show the direct one. Let :

$$y \in [S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[= \bigsqcup_{T_L^k(x)=x_n} \left[S_R^{n+k}(x, c(x)); S_R^{n+k}(x, d(x))\right[$$

There is an unique x such that $T_L^k(x) = x_n$ and $y \in \left[S_R^{n+k}(x, c(x)); S_R^{n+k}(x, d(x))\right[$. But then :

$$\left[S_R^{n+k}(x_{n+k}, c(x_{n+k})); S_R^{n+k}(x_{n+k}, d(x_{n+k}))\right[\cap \left[S_R^{n+k}(x, c(x)); S_R^{n+k}(x, d(x))\right[\neq \emptyset$$

which can only happen for $x_{n+k} = x$. Hence $T_L^k(x_{n+k}) = T_L^k(x) = x_n$ and we are done. ■

For any $x' \in \mathbb{S}^1$, let :

$$\Delta_n(x') = \{c(x')\} \cup \{S_R^n(x, d(x)) \mid T_L^n(x) = x'\}$$

Lemma 2.5. *For every $n \geq 0$ and $x' \in \mathbb{S}^1$,*
(i) $\Delta_n(x')$ is finite ;
(ii) $\Delta_n(x') \subset [c(x'); d(x')]$;
(iii) $\Delta_n(x') = \{S_R^n(x, c(x)) \mid T_L^n(x) = x'\} \cup \{d(x')\}$;
(iv) $\Delta_n(x') \subset \Delta_{n+1}(x')$.

PROOF : Since T_L preserves a finite Markov partition and that its branches are homeomorphisms, it can only have a bounded number of preimages, which proves (i). Items (ii) and (iii) are clear thanks to lemma 2.1. We only have to show (iv).

Let $y' = S_R^n(x, d(x)) \in \Delta_n(x')$, with $T_L^n(x) = x'$. Following 2.1,

$$[c(x); d(x)[= \bigsqcup_{T_L(\tilde{x})=x} [S_R(\tilde{x}, c(\tilde{x})); S_R(\tilde{x}, d(\tilde{x}))[$$

$d(x)$ is realized as the right endpoint of one and exactly one sub-intervals of the partition, i.e. there is an unique $\tilde{x}$ such that $T_L(\tilde{x}) = x$ and $S_R(\tilde{x}, d(\tilde{x})) = d(x)$. Hence $T_L^{n+1}(\tilde{x}) = T_L^n(x) = x'$ and :

$$S_R^{n+1}(\tilde{x}, d(\tilde{x})) = S_R^n(T_L(\tilde{x}), S_R(\tilde{x}, d(\tilde{x}))) = S_R^n(x, d(x)) = y' \qquad ■$$

For every $x' \in \mathbb{S}^1$, let :

$$\Delta(x') = \bigcup_{n\geq 0} \Delta_n(x')$$

Lemma 2.6. *Suppose that T_R is expansive. Then $\Delta(x')$ is dense in J_k^R for every $x' \in I_k^L$.*

PROOF : Assume that there is an open interval $U \subset J_k^R = [c(x'); d(x')[$ that does not intersect any of the $\Delta_n(x')$. Since the intervals $[S_R^n(x, c(x)); S_R^n(x, d(x))[$ form a partition of $[c(x'); d(x')[$ when x goes through all T_L^n-preimages of x', and that U does not meet the end of any of them, there is exactly one $x_n \in \mathbb{S}^1$ for every $n \geq 0$ such that :

$$x' = T_L^n(x_n) \qquad \text{and} \qquad U \subset [S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[$$

We will denote by l_n the unique index such that $x_n \in I_{l_n}^L$.

Let $\gamma_n = S_R^n(x_n, c(x_n))$ and $\delta_n = S_R^n(x_n, d(x_n))$. Since $\Delta_n(x') \subset \Delta_{n+1}(x')$, we have :

$$\forall n \geq 0, U \subset [\gamma_{n+1}; \delta_{n+1}[\subset [\gamma_n; \delta_n[\subset J_k^R$$

This means that when J_k^R is lifted to an interval of the real line, the sequence $(\gamma_n)_{n\geq 0}$ is non-decreasing and bounded for above, while the sequence $(\delta_n)_{n\geq 0}$ is non-increasing and bounded from below. As such, they both converge in $\overline{J_k^R}$ to respectively u and v, and :

$$\forall n \geq 0, U \subset]u;v[\subset]\gamma_n;\delta_n[$$

Since there is no point of $\Delta_n(x')$ between γ_n and δ_n, this ensures that $]u;v[$ does not meet $\Delta(x')$. Moreover, u and v being accumulation points of $\Delta(x')$, we have that any open interval of J_k^R that meets $]u;v[$ is either a subset of it or meets $\Delta(x')$.

Fix $n \geq 0$ and let $y' \in [\gamma_n;\delta_n[\subset [S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[$. Because $S_R^n(x_n, .)$ is an homeomorphism from $[c(x_n); d(x_n)[$ onto $[S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[$, there is an unique $y_n \in [c(x_n); d(x_n)[$ such that $y' = S_R^n(x_n, y_n)$. Now since $(x_n, y_n) \in C$, we get by applying T_R^n that :

$$y_n = T_R^n(S_R^n(x_n, y_n)) = T_R^n(y')$$

Hence T_R^n induces an homeomorphism from $[S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))[$ onto $[c(x_n); d(x_n)[$ as the reciprocal function of the homeomorphism $S_R^n(x_n, .)$. In particular, T_R^n is an homeomorphism from $]u;v[$ onto $T_R^n(]u;v[)$, which is then an open interval of $J_{l_n}^R$.

Let $U_n = T_R^n(]u;v[) =]u_n;v_n[\subset J_{l_n}^R$. Be careful that this notation does not mean that $u_n = T_R^n(u)$ or $v_n = T_R^n(v)$. We shall prove that U_n does not meet $\Delta(x_n)$, but that any open interval of $J_{l_n}^R$ that meets U_n is either a subset of it or meets $\Delta(x_n)$.

First assume that there is some $m \geq 0$ such that $U_n \cap \Delta_m(x_n) \neq \emptyset$. Then there is a $\tilde{x} \in \mathbb{S}^1$ such that $T_L^m(\tilde{x}) = x_n$ and $\tilde{y} = S_R^m(\tilde{x}, c(\tilde{x})) \in U_n$. But $T_L^{n+m}(\tilde{x}) = x'$ and we get :

$$S_R^{n+m}(\tilde{x}, c(\tilde{x})) = S_R^n(T_L^m(\tilde{x}), S_R^m(\tilde{x}, c(\tilde{x}))) = S_R^n(x_n, \tilde{y})$$

where $\tilde{y} \in U_n$. Since $S_R^n(x_n, .)$ is the reciprocal of T_R^n over $[c(x_n); d(x_n)[\supset U_n$, we get that $S_R^{n+m}(\tilde{x}, c(\tilde{x})) \in]u;v[$, which contradicts the fact that $]u;v[$ does not meet $\Delta(x')$.

For the second point, let V be an open interval of $J_{l_n}^R$ that meets U_n but is not a subset of it. This means that either u_n or v_n is in V but is not an endpoint of $J_{l_n}^R = [c(x_n); d(x_n)[$. Suppose for example that this is the case for u_n. Since T_R^n preserves orientation, $S_R^n(x_n, V)$ is an open neighbourhood of u in J_k^R But both of these points are accumulation points of $\Delta(x')$, so there is a $y' \in \Delta(x') \cap]\gamma_n;\delta_n[\cap S_R^n(x_n, V)$. On one hand, $y' \in]\gamma_n;\delta_n[$ hence there is $y_n \in [c(x_n); d(x_n)[= J_{l_n}^R$ such that $y' = S_R^n(x_n, y_n)$. Moreover, T_R^n is the reciprocal of $S_R^n(x_n, .)$ on $[\gamma_n;\delta_n[$, so $y_n = T_R^n(y') \in V$. On the other hand, $y' \in]\gamma_n;\delta_n[$ means that $y' \notin \Delta_n(x')$, and as such $y' \in \Delta_{n+m}(x')$ for some $m > 0$. In other words, there is a $\tilde{x} \in \mathbb{S}^1$ such that $T_L^{n+m}(\tilde{x}) = x'$ and $S_R^{n+m}(\tilde{x}, c(\tilde{x})) = y'$. Hence we have :

$$T_C^n(x_n, y_n) = (T_L^n(x_n), S_R^n(x_n, y_n)) = (x', y') = (T_L^{n+m}(\tilde{x}), S_R^{n+m}(\tilde{x}, c(\tilde{x}))) = T_C^{n+m}(\tilde{x}, c(\tilde{x}))$$

Using that T_C is a bijection, we get that :

$$(x_n, y_n) = T_C^m(\tilde{x}, c(\tilde{x})) = (T_l^m(\tilde{x}), S_R^m(\tilde{x}, c(\tilde{x})))$$

which shows that $y_n \in \Delta_m(x_n) \cap V$.

We now consider the collection of open intervals $(U_n)_{n\geq 0}$. Each U_n is a subset of $J_{l_n}^R$. Suppose that there is a fiber J_m^R and $p < q$ such that $l_p = l_q = m$ and $U_p \cap U_q \neq \emptyset$. Since both of these intervals do not meet $\Delta(x_p) = \Delta(x_q)$, we get that $U_p = U_q$. Let $n = q - p > 0$. Then T_R^n induces an homeomorphism from U_p to itself. But T_R is expansive, hence so is T_R^n and T_{R/U_p}^n. What we obtain is an expansive homeomorphism of an interval that preserves orientation and fixes its endpoints, or in other words an expansive homeomorphism of the circle. This is impossible according to Walters. Thus the U_n intervals that lie in the same fiber are always disjoint (although they can intersect an interval from another fiber).

Let $y_1, y_2 \in U_0$. By expansivity, there is a $n_1 > 0$ such that :

$$d(u_{n_1}, v_{n_1}) \geq d(T_R^{n_1}(y_1), T_R^{n_1}(y_2)) > \varepsilon > 0$$

where ε is the expansivity constant of T_R. Repeating this argument by replacing U_0 by U_{n_1+1}, $U_{n_1+1+n_2+1}$ and so on, we get that infinitely many U_n have a length span strictly greater than ε.

But there is only a finite number of fibers, so one of them must contain an infinite number of U_n intervals of length greater than ε. This is impossible since each fiber has bounded length. ■

If $n \geq 0$ and $(x', y') \in C$, we have seen that there is an unique $x \in \mathbb{S}^1$ such that $T_L^n(x) = x'$ and $y' \in [S_R^n(x, c(x)); S_R^n(x, d(x))[$. For this same x, we call $\gamma_n(x', y') = S_R^n(x, c(x))$ the *projection of* y' *on* $\Delta_n(x')$. We also extend γ_n to $\tilde{C}$ by choosing the convention $\gamma_n(x', d(x')) = d(x')$.

Lemma 2.7. *For every* $n \geq 0$ *and* $(x', y') \in \tilde{C}$, $\gamma_{n+1}(x', y') \in [\gamma_n(x', y'); y']$.

PROOF : If $y' = d(x')$, then $\gamma_{n+1}(x', y') = d(x') = \gamma_n(x', y')$ and we're done. Otherwise, y' lies at the same time in $[S_R^n(x, c(x)); S_R^n(x, d(x))[$ (where $\gamma_n(x', y') = S_R^n(x, c(x))$) and in $\left[S_R^{n+1}(w, c(w)); S_R^{n+1}(w, d(w))\right[$ (where $\gamma_{n+1}(x', y') = S_R^{n+1}(w, c(w))$). Hence both of these intervals intersect, and lemma 2.3 gives us that necessarily $T_L(w) = x$ and :

$$y' \in \left[S_R^{n+1}(w, c(w)); S_R^{n+1}(w, d(w))\right[\subset [S_R^n(x, c(x)); S_R^n(x, d(x))[$$

In particular, we have $[\gamma_{n+1}(x', y'); y'] \subset [\gamma_n(x', y'); y']$, which proves the lemma. ■

Lemma 2.8. *For every* $(x', y') \in \tilde{C}$, *the sequence* $(\gamma_n(x', y'))_{n \geq 0}$ *converges to* y'.

PROOF : According to lemma 2.7, this sequence is monotonous. Moreover, the density of $\Delta(x')$ in $[c(x'); d(x')]$ ensures that one can find points of the form $S_R^n(x, c(x))$ (with $T_L^n(x) = x'$) in $[c(x'); y']$ that are arbitrarily close from y'. Hence the sequence converges to y'. ■

2.2. **Extension of an eigenfunction of the Ruelle operator.** In this section, ψ will denote an element of $E_L(1)$. If $n \geq 0$ and $(x', y') \in \tilde{C}$, we define :

$$g_n(x', y') = \sum_{T_L^n(x) = x'} \psi(x) e^{A_L^n(x)} \mathbb{1}_{]c(x'); y']}(S_R^n(x, d(x)))$$

Note that $\mathbb{1}_{]c(x'); y']}(S_R^n(x, d(x)))$ is a fancy way to say "1 if $[S_R^n(x, c(x)); S_R^n(x, d(x))[$ is a subset of $[c(x'); y'[$ and 0 otherwise".

Lemma 2.9. *For every* $n \geq 0$ *and* $x' \in \mathbb{S}^1$, $g_n(x', c(x')) = 0$ *and* $g_n(x', d(x')) = \psi(x')$.

PROOF : If $y' = c(x')$, then the interval $]c(x'); y']$ is empty so the sum evaluates to 0. On the other hand, if $y' = d(x')$ then for every x which is a T_L^n-preimage of x' we have that $S_R^n(x, d(x)) \in]c(x'); y']$. Thus the characteristic function always evaluates to 1 and :

$$g_n(x', d(x')) = \sum_{T_L^n(x) = x'} \psi(x) e^{A_L^n(x)} = \psi(x')$$

using that ψ is an eigenfunction of the Ruelle operator. ■

Lemma 2.10.

$$\forall n \geq 0, \forall x' \in \mathbb{S}^1, \forall y' \in \Delta_n(x'), g_n(x', y') = g_{n+1}(x', y')$$

PROOF : Fix $n \geq 0$ and $x' \in \mathbb{S}^1$. If $y' = c(x')$, $g_n(x', c(x')) = 0 = g_{n+1}(x', c(x'))$. Otherwise, let $y' \in \Delta_n(x') \setminus \{c(x')\}$. One has :

$$\begin{aligned}
g_{n+1}(x', y') &= \sum_{T_L^{n+1}(x) = x'} \psi(x) e^{A_L^{n+1}(x)} \mathbb{1}_{]c(x'); y']}(S_R^{n+1}(x, d(x))) \\
&= \sum_{T_L^n(w) = x'} \sum_{T_L(x) = w} \psi(x) e^{A_L^n(T_L(x)) + A_L(x)} \mathbb{1}_{]c(x'); y']}(S_R^n(T_L(x), S_R(x, d(x)))) \\
&= \sum_{T_L^n(w) = x'} e^{A_L^n(w)} \sum_{T_L(x) = w} \psi(x) e^{A_L(x)} \mathbb{1}_{]c(x'); y']}(S_R^n(w, S_R(x, d(x))))
\end{aligned}$$

But lemma 2.1 tells us that :

$$]c(x'); d(x')] = \bigsqcup_{T_L^n(w) = x'}]S_R^n(w, c(w)); S_R^n(w, d(w))]$$

We can assume that the set $\{w \mid T_L^n(w) = x'\} = \{(w_j)\}$ of the T_L^n-preimages of x' is ordered in such a way that $S_R^n(w_j, d(w_j)) = S_R^n(w_{j+1}, c(w_{j+1}))$. Since $y' \in \Delta_n(x') \setminus \{c(x')\}$, there is a k such that $y' = S_R^n(w_k, d(w_k))$, and then :

$$]c(x'); y'] = \bigsqcup_{j<k}]S_R^n(w_j, c(w_j)); S_R^n(w_j, d(w_j))]$$

Now let's have a look when w is fixed : $\{S_R(x, d(x)) \mid T_L(x) = w\} \subset]c(w); d(w)]$. There are two alternatives :

– Either $w = w_j$ with $j < k$, and for every x such that $T_L(x) = w$,
$$S_R^n(w, S_R(x, d(x))) \in]S_R^n(w_j, c(w_j)); S_R^n(w_j, d(w_j))] \subset]c(x'); y'] \,;$$
– Or $w = w_j$ with $j \geq k$, and for every x such that $T_L(x) = w$,
$$S_R^n(w, S_R(x, d(x))) \in]S_R^n(w_j, c(w_j)); S_R^n(w_j, d(w_j))] \subset]y'; d(x')]$$
which does not meet $]c(x'); y']$.

This means that for every x such that $T_L(x) = w$, $S_R^n(w, S_R(x, d(x))) \in]c(x'); y']$ if and only if $w = w_j$ with $j < k$, i.e. if and only if $S_R^n(w, d(w)) \in]c(x'); y']$.

Consequently,

$$\begin{aligned} g_{n+1}(x', y') &= \sum_{T_L^n(w)=x'} e^{A_L^n(w)} \left(\sum_{T_L(x)=w} \psi(x) e^{A_L(x)} \right) \mathbb{1}_{]c(x');y']}(S_R^n(w, d(w))) \\ &= \sum_{T_L^n(w)=x'} \psi(w) e^{A_L^n(w)} \mathbb{1}_{]c(x');y']}(S_R^n(w, d(w))) \\ &= g_n(x', y') \end{aligned}$$

■

Lemma 2.11.

$$\forall n \geq 0, \forall (x, y) \in \tilde{C}, g_{n+1}(T_C(x, y)) = g_n(x, y) e^{A_L(x)} + g_1(T_C(x, c(x)))$$

PROOF : Fix $n \geq 0$ and $(x', y') \in \tilde{C}$. If $y' = c(x')$, $g_n(x', c(x')) = 0$ but since $S_R(x', c(x')) \in \Delta_1(T_L(x'))$ we get thanks to lemma 2.10 that :

$$g_1(T_C(x', c(x'))) = g_2(T_C(x', c(x'))) = \ldots = g_{n+1}(T_C(x', c(x')))$$

so the relation holds.

We now assume that $y' \in]c(x'); d(x')]$. One has :

$$\begin{aligned} g_{n+1}(T_C(x', y')) &= g_{n+1}(T_L(x'), S_R(x', y')) \\ &= \sum_{T_L^{n+1}(x)=T_L(x')} \psi(x) e^{A_L^{n+1}(x)} \mathbb{1}_{]c(T_L(x'));S_R(x',y')]}(S_R^{n+1}(x, d(x))) \\ &= \sum_{T_L(w)=T_L(x')} e^{A_L(w)} \\ &\qquad \sum_{T_L^n(x)=w} \psi(x) e^{A_L^n(x)} \mathbb{1}_{]c(T_L(x'));S_R(x',y')]}(S_R(w, S_R^n(x, d(x)))) \end{aligned}$$

Recall that if $\{w_0 \ldots w_{p-1}\}$ are the T_L-preimages of $T_L(x')$, then :

$$]c(T_L(x')); d(T_L(x'))] = \bigsqcup_{i=0}^{p-1}]S_R(w_i, c(w_i)); S_R(w_i, d(w_i))]$$

and we can assume that they are ordered in such a way that $S_R(w_i, d(w_i)) = S_R(w_{i+1}, c(w_{i+1}))$. Moreover, there is an unique k such that $x' = w_k$. Since $y' \in]c(x'); d(x')]$,

$$S_R(x', y') \in]S_R(x', c(x')); S_R(x', d(x'))] =]S_R(w_k, c(w_k)); S_R(w_k, d(w_k))]$$

Consequently,

$$]c(T_L(x')); S_R(x', y')] = \bigsqcup_{i=0}^{k-1}]S_R(w_i, c(w_i); S_R(w_i, d(w_i))] \sqcup]S_R(x', c(x')); S_R(x', y')]$$

Now suppose that $T_L^n(x) = w_i$ for some i, then $S_R^n(x, d(x)) \in]c(w_i); d(w_i)]$ and :

$$S_R(w_i, S_R^n(x, d(x))) \in]S_R(w_i, c(w_i)); S_R(w_i, d(w_i))]$$

Thus $S_R(w, S_R^n(x, d(x))) \in]c(T_L(x')); S_R(x', y')]$ if and only if :

– Either $w = w_j$ with $j < k$, in which case we necessarily have :

$$S_R(w, S_R^n(x, d(x))) \in]S_R(w_j, c(w_j)); S_R(w_j, d(w_j))] \subset]c(T_L(x')); S_R(x', y')] ;$$

– Or $w = w_k = x'$, in which case x must satisfy :

$$S_R(x', S_R^n(x, d(x))) \in]S_R(x', c(x')); S_R(x', y')] \Leftrightarrow S_R^n(x, d(x)) \in]c(x'); y']$$

Using that ψ is an eigenfunction of the Ruelle operator, we get :

$$\begin{aligned} g_{n+1}(T_C(x', y')) &= e^{A_L(x')} \sum_{T_L^n(x)=x'} \psi(x) e^{A_L^n(x)} \mathbb{1}_{]c(x');y']}(S_R^n(x, d(x))) \\ &\quad + \sum_{i=0}^{k-1} e^{A_L(w_i)} \sum_{T_L^n(x)=w_i} \psi(x) e^{A_L^n(x)} \\ &= e^{A_L(x')} g_n(x', y') + \sum_{i=0}^{k-1} \psi(w_i) e^{A_L(w_i)} \end{aligned}$$

But $w = w_j$ with $j < k$ if and only if :

$$]S_R(w, c(w)); S_R(w, d(w))] \subset \bigsqcup_{i=0}^{k-1}]S_R(w_i, c(w_i)); S_R(w_i, d(w_i))]$$

i.e. if and only if :

$$S_R(w, d(w)) \in]S_R(w_0, c(w_0)); S_R(w_{k-1}, d(w_{k-1}))] =]c(T_L(x')); S_R(x', c(x'))]$$

Hence :

$$\begin{aligned} g_{n+1}(T_C(x', y')) &= e^{A_L(x')} g_n(x', y') \\ &\quad + \sum_{T_L(w)=T_L(x')} \psi(w) e^{A_L(w)} \mathbb{1}_{]c(T_L(x'));S_R(x',c(x'))]}(S_R(w, d(w))) \\ &= e^{A_L(x')} g_n(x', y') + g_1(T_C(x', c(x'))) \end{aligned}$$

■

2.3. **Proof of the injectivity.**

Lemma 2.12. *Let* $[a; b[\subset \mathbb{S}^1$ *and* $c \in [a; b]$. *Then for every absolutely continuous map* $\varphi : [a; b] \to \mathbb{C}$ *we have :*

$$\langle \nu, \varphi \mathbb{1}_{[a;b[} \rangle = \langle \nu, \varphi \mathbb{1}_{[a;c[} \rangle + \langle \nu, \varphi \mathbb{1}_{[c;b[} \rangle$$

Lemma 3.1. *Let* I *be a non-empty half-open interval of* $\mathbb{S}^1$, $\gamma : \overline{I} \to \gamma(\overline{I})$ *a* $\mathcal{C}^1$*-diffeomorphism,* $h : \mathbb{S}^1 \to \mathbb{C}$ *a continuous map over* $\overline{I} \cup \gamma(\overline{I})$*, and* $f : I \to \mathbb{C}$ *an absolutely continuous map over* $\overline{I}$. *Let* ν *be the linear operator given by the weak derivative of* h. *If for every* $\varphi : I \to \mathbb{C}$ *absolutely continuous on* $\overline{I}$ *:*

$$\left\langle \nu, (\varphi f) \circ \gamma^{-1} \mathbb{1}_{\gamma(I)} \right\rangle = \langle \nu, \varphi \mathbb{1}_I \rangle$$

then for every half-open sub-inteval J *of* I *and every* $\varphi : J \to \mathbb{C}$ *absolutely continuous on* $\overline{J}$ *:*

$$\left\langle \nu, (\varphi f) \circ \gamma^{-1} \mathbb{1}_{\gamma(J)} \right\rangle = \langle \nu, \varphi \mathbb{1}_J \rangle$$

Lemma 2.13. *Let $n \geq 0$, $I \in \mathcal{P}^R_{n+1}$ and note $\gamma = T^n_{R/I}$. Then for every absolutely continuous map $\varphi : \overline{I} \to \mathbb{C}$ we have :*

$$\langle \nu, e^{A^n_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1} \mathbb{1}_{\gamma(I)} \rangle = \langle \nu, \varphi \mathbb{1}_I \rangle$$

PROOF : We shall prove the result by induction over $n \geq 0$.

If $n = 0$, then $I = I^R_k \in \mathcal{P}^R_1$ for some k, $\gamma = T^0_{R/I} = \text{id}$, $A^0_R = 0$ and we have indeed that for every map φ absolutely continuous over $\overline{I}$:

$$\langle \nu, e^{A^0_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1} \mathbb{1}_{\gamma(I)} \rangle = \langle \nu, \varphi \mathbb{1}_I \rangle$$

Now assume that the result is verified for all cylinders of length n. Let $I \in \mathcal{P}^R_{n+1}$ and φ absolutely continuous over $\overline{I}$. There is a $k \in [\![1\,;\,N]\!]$ such that $I \subset I^R_k$. Let $\gamma = T_{R/I^R_k}$, so that $T_{R/I} = \gamma_{/I}$. Since γ is a $\mathcal{C}^1$-diffeomorphism from $\overline{I^R_k}$ to $\gamma(\overline{I^R_k})$, any $y \in \mathbb{S}^1$ may have at most one T_R-preimage in I^R_k, and this happens if and only if $y \in \gamma(I^R_k)$, hence we have :

$$\mathcal{L}_R \left[\varphi \mathbb{1}_{I^R_k}\right] = e^{A_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1} \mathbb{1}_{\gamma(I^R_k)}$$

Using that ν is an eigendistribution of $\mathcal{L}_R$, we get that :

$$\langle \nu, e^{A_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1} \mathbb{1}_{\gamma(I^R_k)} \rangle = \langle \nu, \mathcal{L}_R \left[\varphi \mathbb{1}_{I^R_k}\right] \rangle = \langle \nu, \varphi \mathbb{1}_{I^R_k} \rangle$$

which, according to lemma 3.1 applied for $J = I \subset I^R_k$, $f = e^{A_R}$ and the same γ, implies that :

$$\langle \nu, e^{A_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1} \mathbb{1}_{\gamma(I)} \rangle = \langle \nu, \varphi \mathbb{1}_I \rangle$$

But $\tilde{I} = T_R(I) = \gamma(I)$ is a cylinder of length n, so if we note $\tilde{\gamma} = T^{n-1}_R{}_{/\tilde{I}}$ then the induction process applied for the cylinder $\tilde{I}$ and the map $e^{A_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1}$ (which is absolutely continuous over $\overline{\tilde{I}}$) yields :

$$\langle \nu, e^{A^{n-1}_R \circ \tilde{\gamma}^{-1}} e^{A_R \circ (\gamma^{-1} \circ \tilde{\gamma}^{-1})} \varphi \circ (\gamma^{-1} \circ \tilde{\gamma}^{-1}) \mathbb{1}_{\tilde{\gamma}(\tilde{I})} \rangle = \langle \nu, e^{A_R \circ \gamma^{-1}} \varphi \circ \gamma^{-1} \mathbb{1}_{\tilde{I}} \rangle = \langle \nu, \varphi \mathbb{1}_I \rangle$$

which is exactly the desired relation since $\tilde{\gamma}\gamma = T^n_{R/I}$ and $\tilde{\gamma}(\tilde{I}) = T^n_R(I)$. ■

Lemma 2.14. *Let $x' \in \mathbb{S}^1$, $n \geq 0$ and x such that $T^n_L(x) = x'$. There exist $I_1, \ldots, I_m \in \mathcal{P}^R_{n+1}$ such that :*

$$[S^n_R(x, c(x)); S^n_R(x, d(x))[= \bigsqcup_{j=1}^{m} I_j$$

PROOF : The set of endpoints $\partial \mathcal{P}^R_{n+1}$ of all the intervals from $\mathcal{P}^R_{n+1}$ are precisely the T^n_R-preimages of the partition points of $\mathcal{P}^R$. As $S^n_R(x, c(x))$ and $S^n_R(x, d(x))$ are two such points ($c(x)$ and $d(x)$ have been supposed to be in $\mathcal{P}^R$), the result follows. ■

Lemma 2.15. *Let $\nu \in F_R(1)$ of density h, and let $\psi = \Phi(\nu)$. Let $x' \in \mathbb{S}^1$ such that $y' \mapsto W(x', y')$ is absolutely continuous over $J^R(x')$. Then for every $n \geq 0$ and x such that $T^n_L(x) = x'$ and for which $y \mapsto W(x, y)$ is absolutely continuous over $J^R(x)$, we have :*

$$\psi(x) e^{A^n_L(x)} = e^{W(x', S^n_R(x, d(x)))} h(S^n_R(x, d(x))) - e^{W(x', S^n_R(x, c(x)))} h(S^n_R(x, c(x)))$$
$$- \int_{S^n_R(x, c(x))}^{S^n_R(x, d(x))} \left[\partial_2 e^W\right](x', t) h(t) dt$$

PROOF : For one such x, lemma 2.14 tells us that $[S^n_R(x, c(x)); S^n_R(x, d(x))[$ is the disjoint and contiguous union of T_R-cylinders $I_1, \ldots, I_m$ of length $n + 1$. Remark that since T^n_R induces an $\mathcal{C}^1$-diffeomorphism from $[S^n_R(x, c(x)); S^n_R(x, d(x))]$ onto $[c(x); d(x)]$, then the branch $\gamma = T^n_{R/\left[S^n_R(x,c(x)); S^n_R(x,d(x))\right]}$ is such that $T^n_{R/I_j} = \gamma_{/I_j}$ for every j. Now apply lemma 2.13 with $\varphi = e^{W(x', .)}$ and $I = I_j$ to get :

$$\forall j \in [\![1\,;\,m]\!], \langle \nu, e^{A^n_R(\gamma^{-1}(.)) + W(x', \gamma^{-1}(.))} \mathbb{1}_{\gamma(I_j)}(.) \rangle = \langle \nu, e^{W(x', .)} \mathbb{1}_{I_j}(.) \rangle$$

Since the I_j are disjoint, contiguous and cover $[S_R^n(x,c(x));S_R^n(x,d(x))[$, their images $\gamma(I_j)$ by the $\mathcal{C}^1$-diffeomorphism γ are also disjoint, contiguous and cover $[c(x);d(x)[$. Hence summing all those equalities for j between 1 and m yields :

$$\langle \nu, e^{A_R^n(\gamma^{-1}(.))+W(x',\gamma^{-1}(.))}\mathbb{1}_{[c(x);d(x)[}(.)\rangle = \langle \nu, e^{W(x',.)}\mathbb{1}_{[S_R^n(x,c(x));S_R^n(x,d(x))[}(.)\rangle$$

Noting that the inverse of γ is given by $\gamma^{-1}(y) = S_R^n(x,y)$, and using the definition of W, we get that :

$$\forall y \in [c(x);d(x)[\,, A_R^n(S_R^n(x,y)) + W(x',S_R^n(x,y)) = W(x,y) + A_L^n(x)$$

which simplifies the previous relation into :

$$e^{A_L^n(x)}\langle \nu, e^{W(x,.)}\mathbb{1}_{[c(x);d(x)[}(.)\rangle = \langle \nu, e^{W(x',.)}\mathbb{1}_{[S_R^n(x,c(x));S_R^n(x,d(x))[}(.)\rangle$$

We then recognize $\psi(x)$ on the left hand side of the equality, and the desired quantity on its right hand side. ■

Lemma 2.16.

$$g_n(x',y') = e^{W(x',\gamma_n(x',y'))}h(\gamma_n(x',y')) - e^{W(x',c(x'))}h(c(x')) - \int_{c(x')}^{\gamma_n(x',y')} \left[\partial_2 e^W\right](x',t)h(t)dt$$

PROOF : This is straightforward by noting that $g_n(x',y') = g_n(x',\gamma_n(x',y'))$ and by summing the relations given by lemma 2.15. ■

Theorem 2.17. *The map Φ is injective.*

PROOF : Assume that $\nu = h' \in F_R(1)$ is such that $\psi = \Phi(\nu) = 0$. Then the corresponding sequence of functions g_n is always 0 for every n, and lemma 2.16 implies that for every $y' \in [c(x');d(x')]$ we have :

$$e^{W(x',\gamma_n(x',y'))}h(\gamma_n(x',y')) - e^{W(x',c(x'))}h(c(x')) = \int_{c(x')}^{\gamma_n(x',y')} \left[\partial_2 e^W\right](x',t)h(t)dt$$

However, h is continuous and by lemma 2.8 $(\gamma_n(x',y'))_{n\geq 0}$ converges to y' when n goes to the infinity, so by taking the limit we get :

$$e^{W(x',y')}h(y') - e^{W(x',c(x'))}h(c(x')) = \int_{c(x')}^{y'} \left[\partial_2 e^W\right](x',t)h(t)dt$$

This means that the map $y' \mapsto e^{W(x',y')}h(y')$ is absolutely continuous over $[c(x');d(x')]$ with derivative Lebesgue almost everywhere equal to :

$$\left[e^{W(x',.)}h(.)\right]'(y') = \left[e^{W(x',.)}\right]'(y')h(y')$$

Then h is also absolutely continuous over $[c(x');d(x')]$ and its derivative is Lebesgue almost everywhere equal to :

$$\begin{aligned} h'(y') &= \left[e^{W(x',.)}h(.)\right]'(y')e^{-W(x',y')} + e^{W(x',y')}h(y')\left[e^{-W(x',.)}\right]'(y') \\ &= \left[e^{W(x',.)}\right]'(y')h(y')e^{-W(x',y')} + e^{W(x',y')}h(y')\left[e^{-W(x',.)}\right]'(y') \\ &= \left[e^{W(x',.)-W(x',.)}\right]'(y')h(y') = 0 \end{aligned}$$

Hence h is constant everywhere and $\nu = 0$. ■

2.4. **Proof of the surjectivity.** For any $x \in \mathbb{S}^1$, we note $\eta(x) = g_1(T_C(x, c(x)))$.

Lemma 2.18. *Let $n \geq 0$ and $(x, y) \in C$. Then :*

$$g_n(T_C^n(x,y)) = \sum_{k=0}^{n-1} \eta(T_L^k(x)) e^{A_L^{n-1-k}(T_L^k(x))}$$

PROOF : We prove this result by induction over $n \geq 0$.

Suppose $n = 0$. Since $y \neq d(x)$, $d(x) \notin]c(x); y]$ and we have $g_0(x, y) = 0$. On the other hand the sum is empty, so the relation holds.

Now assume that the relation is satisfied for some $n \geq 0$. Then lemma 2.11 gives us that :

$$\begin{aligned} g_{n+1}(T_C^{n+1}(x,y)) &= g_n(T_C^n(x,y)) e^{A_L(T_L^n(x))} + \eta(T_L^n(x)) \\ &= \sum_{k=0}^{n-1} \eta(T_L^k(x)) e^{A_L(T_L^n(x)) + A_L^{n-1-k}(T_L^k(x))} + \eta(T_L^n(x)) \\ &= \sum_{k=0}^{n-1} \eta(T_L^k(x)) e^{A_L(T_L^{n-k}(T_L^k(x))) + A_L^{n-1-k}(T_L^k(x))} + \eta(T_L^n(x)) e^{A_L^0(T_L^n(x))} \\ &= \sum_{k=0}^{n} \eta(T_L^k(x)) e^{A_L^{n-k}(T_L^k(x))} \end{aligned}$$

■

Lemma 2.19. *Let $n \geq 0$ and $(x', y') \in C$. Then :*

$$g_n(x', y') = \sum_{k=0}^{n-1} \eta(S_L^{k+1}(x', y')) e^{A_L^k(S_L^{k+1}(x', y'))}$$

PROOF : Let $(x, y) = T_C^{-n}(x', y')$, so that $x = S_L^n(x', y')$. Recall that for every $k \leq n$:

$$T_L^k(x) = T_L^k(S_L^n(x', y')) = S_L^{n-k}(x', y')$$

Applying lemma 2.18, we get :

$$\begin{aligned} g_n(x', y') = g_n(T_L^n(x, y)) &= \sum_{k=0}^{n-1} \eta(T_L^k(x)) e^{A_L^{n-1-k}(T_L^k(x))} \\ &= \sum_{k=0}^{n-1} \eta(S_L^{n-k}(x', y')) e^{A_L^{n-1-k}(S_L^{n-k}(x', y'))} \\ &= \sum_{k=0}^{n-1} \eta(S_L^{k+1}(x', y')) e^{A_L^k(S_L^{k+1}(x', y'))} \end{aligned}$$

■

Lemma 2.20. *The set of points $(x', y') \in C$ such that :*

$$\sum_{k \geq 0} e^{\Re A_L^k(S_L^k(x', y'))} < \infty$$

is T_C-invariant.

PROOF : Let $(x', y') \in C$ and $(x'', y'') = T_C(x', y')$, so that $x' = S_L(x'', y'')$ and $y' = T_R(y'')$. Pick $k \geq 0$. Note that :

$$S_L^{k+1}(x'', y'') = S_L^k(S_L(x'', y''), T_R(y'')) = S_L^k(x', y')$$

Hence :

$$e^{A_L^{k+1}(S_L^{k+1}(x'', y''))} = e^{A_L^{k+1}(S_L^k(x', y'))} = e^{A_L(T_L^k(S_L^k(x', y')))} e^{A_L^k(S_L^k(x', y'))} = e^{A_L(x')} e^{A_L^k(S_L^k(x', y'))}$$

But since A_L is continuous on the closure of each interval of the Markov partition, one can find $K_1, K_2 \geq 0$ such that :

$$\forall x \in \mathbb{S}^1, K_1 \leq e^{\Re A_L(x)} \leq K_2$$

This proves that both series of general term $(e^{\Re A_L^k(S_L^k(x',y'))})_{k\geq 0}$ and $(e^{\Re A_L^k(S_L^k(x'',y''))})_{k\geq 0}$ converge simultaneously. ■

Lemma 2.21. *Suppose that ψ is bounded. Let $(x,y) \in \tilde{C}$. If :*

$$\sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x,y))} < \infty$$

then the sequence $(g_n(x,y))_{n\geq 0}$ converges to some $g(x,y) \in \mathbb{C}$. Moreover, there is a $K \geq 0$ such that :

$$\forall n \geq 0, |g(x,y) - g_n(x,y)| \leq K \sum_{k\geq n+1} e^{\Re A_L^k(S_L^{k+1}(x,y))}$$

PROOF : First note that if ψ is bounded then :

$$\|\eta\|_\infty \leq N \|e^{A_L}\psi\|_\infty = K$$

where N is the number of intervals of the Markov partition, which bounds the number of preimages by T_L. Now if $n, p \geq 0$, then we get by lemma 2.20 that :

$$|g_{n+p}(x,y) - g_n(x,y)| \leq K \sum_{k=n}^{n+p-1} e^{\Re A_L^k(S_L^{k+1}(x,y))}$$

which goes to zero whenever p, n go to the infinity because we have assumed that this series converges absolutely. Hence $g_n(x,y)$ converges to some $g(x,y) \in \mathbb{C}$ and we get the desired inequality by taking the limit when p goes to the infinity. ■

Lemma 2.22. *Suppose that ψ is bounded. Let $(x',y') \in C$ such that :*

$$\sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x',y'))} < \infty$$

Then :

$$g(T_C(x',y')) = g(x',y')e^{A_L(x')} + \eta(x')$$

PROOF : According to lemma 2.21, the hypothesis are enough to ensure that $g(x',y')$ is well defined and that $(g_n(x',y'))_{n\geq 0}$ converges to $g(x',y')$. But lemma 2.20 tells us that the set of points in C that satisfy the summation condition is T_C-invariant, so $g(T_C(x',y'))$ is also well defined and $(g_n(T_C(x',y')))_{n\geq 0}$ converges to $g(T_C(x',y'))$. The functional equation is then obtained by taking the limit when n goes to the infinity in lemma 2.11. ■

Lemma 2.23. *Let $x' \in \mathbb{S}^1$. The function series :*

$$y' \mapsto \sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x',y'))}$$

converges uniformly on $[c(x'); d(x')]$ if and only if the functions series :

$$y \mapsto \sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x,y))}$$

converge uniformly on $[c(x); d(x)]$ for every T_L-preimage x of x'.

PROOF : Recall that the collection :

$$\{[S_R(x, c(x)); S_R(x, d(x))] \mid T_L(x) = x'\}$$

is a finite cover of $[c(x');d(x')]$ by compact intervals. Hence if $q \geq p \geq 1$ are two integers, we have :

$$\begin{aligned}\sup_{y'\in[c(x');d(x')]}\sum_{k=p}^{q} e^{\Re A_L^k(S_L^k(x',y'))} &= \sup_{T_L(x)=x'}\sup_{y\in[c(x);d(x)]}\sum_{k=p}^{q} e^{\Re A_L^k(S_L^k(T_L(x),S_R(x,y)))}\\ &= \sup_{T_L(x)=x'}\sup_{y\in[c(x);d(x)]}\sum_{k=p}^{q} e^{\Re A_L^k(S_L^{k-1}(x,y))}\\ &= \sup_{T_L(x)=x'}\sup_{y\in[c(x);d(x)]}\sum_{k=p-1}^{q-1} e^{\Re A_L^{k+1}(S_L^k(x,y))}\end{aligned}$$

But since there are $K_1, K_2 \geq 0$ such that :

$$\forall x \in \mathbb{S}^1, K_1 \leq e^{\Re A_L} \leq K_2$$

we have that for any $(x,y) \in C$ and $k \geq 0$:

$$K_1 e^{\Re A_L^k(S_L^k(x,y))} \leq e^{\Re A_L^{k+1}(S_L^k(x,y))} \leq K_2 e^{\Re A_L^k(S_L^k(x,y))}$$

hence :

$$\sup_{y'\in[c(x');d(x')]}\sum_{k=p}^{q} e^{\Re A_L^k(S_L^k(x',y'))}$$

converges to 0 when p, q goes to the infinity if and only if :

$$\sup_{T_L(x)=x'}\sup_{y\in[c(x);d(x)]}\sum_{k=p-1}^{q-1} e^{\Re A_L^k S_L^k(x,y))}$$

converges to 0 when p, q goes to the infinity, i.e. if and only if :

$$\sup_{y\in[c(x);d(x)]}\sum_{k=p}^{q} e^{\Re A_L^k S_L^k(x,y))}$$

converges to 0 when p, q goes to the infinity for every x T_L-preimage of x'. ■

Lemma 2.24. *Suppose that $x' \in \mathbb{S}^1$ is such that the function series :*

$$y' \mapsto \sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x',y'))}$$

converges uniformly on $[c(x');d(x')]$. Then :

$$\lim_{n\to+\infty}\sup_{T_L^n(x)=x'} e^{A_L^n(x)} = 0$$

PROOF : Let $x' \in \mathbb{S}^1$ as required. For every point x such that $T_L^n(x) = x'$, any :

$$y' \in [S_R^n(x,c(x)); S_R^n(x,d(x))[\subset [c(x');d(x')[$$

satisfies $x = S_L^n(x',y')$. For one such y', we have :

$$e^{\Re A_L^n(x)} = e^{\Re A_L^n(S_L^n(x',y'))} \leq \sum_{k\geq n} e^{\Re A_L^k(S_L^k(x',y'))}$$

But by hypothesis this remainder converges uniformly to 0 whenever n goes to the infinity, thus for every $\varepsilon > 0$ there is an integer N such that :

$$\forall n \geq N, e^{\Re A_L^n(x)} \leq \sup_{y'\in[c(x');d(x')]}\sum_{k\geq n} e^{\Re A_L^k(S_L^k(x',y'))} \leq \varepsilon$$

which proves the result. ■

Lemma 2.25. *Suppose that ψ is bounded. Let $x' \in \mathbb{S}^1$. If the function series :*

$$y' \in [c(x'); d(x')] \mapsto \sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x',y'))}$$

converges uniformly on $[c(x'); d(x')]$, then the sequence of functions $y' \mapsto g_n(x', y')$ converges uniformly on $[c(x'); d(x')]$ to $y' \mapsto g(x', y')$. Moreover, this limit map is continuous on $[c(x'); d(x')]$.

PROOF : It is clear from lemma 2.21 that $y' \mapsto g_n(x', y')$ converges uniformly to $y' \mapsto g(x', y')$. However, the g_n maps are not continuous. So take $\varepsilon > 0$. By uniform convergence, there exists an integer n_1 such that :

$$\forall n \geq n_1, \forall y' \in [c(x'); d(x')], |g(x', y') - g_n(x', y')| < \varepsilon$$

According to lemma 2.24, there is also an integer n_2 such that :

$$\forall n \geq n_2, \forall x, T_L^n(x) = x' \Rightarrow e^{A_L^n(x)} < \varepsilon$$

Now pick n greater than both n_1 and n_2, and let α be smaller than the diameter of the smallest connected component of $[c(x'); d(x')] \setminus \Delta_n(x')$. With this choice of α, any two points $y_1', y_2' \in [c(x'); d(x')]$ at distance less than α to each other are either in the same connected component of $[c(x'); d(x')] \setminus \Delta_n(x')$, in which case $g_n(x', y_1') = g_n(x', y_2')$, or lie in contiguous components and :

$$|g_n(x', y_1') - g_n(x', y_2')| \leq \|\psi\|_\infty \sup_{T_L^n(x)=x'} e^{\Re A_L^n(x)} < \|\psi\|_\infty \varepsilon$$

In both cases, we have :

$$|g(x', y_1') - g(x', y_2')| \leq (2 + \|\psi\|_\infty)\varepsilon$$

whenever y_1', y_2' are at distance less than α to each other. ■

If $n \geq 0$ and $(x', y') \in \tilde{C}$, we define :

$$h_n(x', y') = e^{-W(x',y')} g_n(x', y') - e^{-W(x',c(x'))} g_n(x', c(x')) - \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x', t) g_n(x', t) dt$$

$$= e^{-W(x',y')} g_n(x', y') - \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x', t) g_n(x', t) dt$$

Thanks to this lemma, we also have a convergence result for the sequence of functions $(h_n)_{n\geq 0}$:

Lemma 2.26. *Assume that ψ is bounded. Let $x' \in \mathbb{S}^1$. We suppose that the function series :*

$$y' \mapsto \sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x',y'))}$$

converges uniformly on $\overline{J^R(x')} = [c(x'); d(x')]$ and that $y' \mapsto W(x', y')$ is absolutely continuous on the same interval. Then the sequence of functions $y' \mapsto h_n(x', y')$ converges uniformly on $\overline{J^R(x')}$ to a map $y' \mapsto h(x', y')$ that satisfies :

$$\forall y' \in [c(x'); d(x')], h(x', y') = e^{-W(x',y')} g(x', y') - \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x', t) g(x', t) dt$$

Moreover, $y' \mapsto h(x', y')$ is continuous on $\overline{J^R(x')}$.

PROOF : Since $y' \mapsto W(x', y')$ is absolutely continuous, then so is $y' \mapsto e^{-W(x',y')}$ (because W is bounded and $\exp$ has bounded variations on compacts). Thus $y' \mapsto e^{-W(x',y')}$ is continuous on $[c(x'); d(x')]$ and its derivative $y' \mapsto \left[\partial_2 e^{-W}\right](x', y')$ is defined Lebesgue almost everywhere and is Lebesgue integrable on $[c(x'); d(x')]$. The convergence and the expression of h is then an immediate consequence of the pointwise convergence of the sequence of functions $y' \mapsto$

$g_n(x',y')$ and Lebegue's dominated convergence theorem. The continuity of $y'\mapsto g(x',y')$ and the properties of $y'\mapsto e^{-W(x',y')}$ then imply the continuity of $y'\mapsto h(x',y')$. ■

Note that we only needed the pointwise convergence of the g_n, but the continuity of g can only be currently obtained by the uniform continuity of said sequence.

It is possible to recover ψ from h pretty easily with the help of this lemma :

Lemma 3.3. *Let $f,g:I\to\mathbb{C}$ continuous maps over $\overline{I}$ and $\varphi:I\to\mathbb{C}$ absolutely continuous over $\overline{I}$ such that :*

$$\forall y,y'\in I, g(y')-g(y)=\varphi(y')f(y')-\varphi(y)f(y)-\int_y^{y'}\varphi'(t)f(t)dt$$

If φ never vanishes on $\overline{I}$, then :

$$\forall y,y'\in I, f(y')-f(y)=\frac{g(y')}{\varphi(y')}-\frac{g(y)}{\varphi(y)}-\int_y^{y'}\frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)g(t)dt$$

Lemma 2.27. *Under the same hypothesis as for lemma 2.26, one has :*

$$\psi(x')=e^{W(x',d(x'))}h(x',d(x'))-e^{W(x',c(x'))}h(x',c(x'))-\int_{c(x')}^{d(x')}\left[\partial_2 e^W\right](x',t)h(x',t)dt$$

PROOF : Note that according to lemma 2.26 we have :

$$\begin{aligned}h(x',d(x'))-h(x',c(x'))&=e^{-W(x',d(x'))}g(x',d(x'))-e^{-W(x',c(x'))}g(x',c(x'))\\&\quad-\int_{c(x')}^{d(x')}\left[\partial_2 e^{-W}\right](x',t)g(x',t)dt\end{aligned}$$

If we apply lemma 3.3 with $y=c(x')$, $y'=d(x')$, $f(t)=g(x',t)$, $g(t)=h(x',t)$ and $\varphi(t)=e^{-W(x',t)}$, we get :

$$\begin{aligned}\psi(x')=g(x',d(x'))-g(x',c(x'))&=e^{W(x',d(x'))}h(x',d(x'))-e^{W(x',c(x'))}h(x',c(x'))\\&\quad-\int_{c(x')}^{d(x')}\left[\partial_2 e^{W}\right](x',t)h(x',t)dt\end{aligned}$$ ■

We shall now prove that $h(x',y')$ essentially does not depend on the choice of $x'\in J^L(y')$. We will first show that $h(x',y')$ only depends on the interval of the Markov partition of T_L in which x' lies (lemma 2.30), then we will express the functional equation of g from lemma 2.22 in terms of h (lemma 2.31), which will give us a way to show that $y'\mapsto h(x'_1,y')-h(x'_2,y')$ is constant on $\overline{J^R(x'_1)}\cap\overline{J^R(x'_2)}$ (lemma 2.34).

Lemma 3.2. *Let $y\in[a;b]$, $c\in\,]a;b]$ and $\varphi:[a;b]\to\mathbb{C}$ absolutely continuous. Then :*

$$\varphi(y)\mathbb{1}_{]a;y]}(c)-\int_a^y\varphi'(t)\mathbb{1}_{]a;t]}(c)dt=\varphi(c)\mathbb{1}_{]a;y]}(c)$$

Lemma 2.28. *Let $n\geq 0$ and $(x',y')\in\tilde{C}$. Then :*

$$h_n(x',y')=\sum_{T_L^n(x)=x'}\psi(x)e^{A_L^n(x)-W(x',S_R^n(x,d(x)))}\mathbb{1}_{]c(x');y']}(S_R^n(x,d(x)))$$

PROOF : Replacing g_n by its definition in the expression of h_n, we get :

$$\begin{aligned}h_n(x',y')=\sum_{T_L^n(x)=x'}\psi(x)e^{A_L^n(x)}\Bigg(&e^{-W(x',y')}\mathbb{1}_{]c(x');y']}(S_R^n(x,d(x)))\\&-\int_{c(x')}^{y'}\left[\partial_2 e^{-W}\right](x',t)\mathbb{1}_{]c(x');t]}(S_R^n(x,d(x)))dt\Bigg)\end{aligned}$$

The desired formula is then obtained by applying lemma 3.2 with $a = c(x')$, $b = d(x')$, $y = y'$, $c = S_R^n(x, d(x))$ and $\varphi(t) = e^{-W(x',t)}$. ■

Lemma 2.29. *Let I_k^L be an interval of the Markov partition of T_L. For $x' \in I_k^L$, the quantity :*

$$\sup_{T_L^n(x)=x'} e^{A_R^n(S_R^n(x,d(x)))}$$

only depends on k and not on the choice of $x' \in I_k^L$.

Moreover, suppose that $x \mapsto W(x, d(x))$ is bounded on $\mathbb{S}^1$, and that there is $x_0' \in \mathbb{S}^1$ is such that the function series :

$$y' \mapsto \sum_{k\geq 0} e^{\Re A_L^k(S_L^k(x_0',y'))}$$

converges uniformly on $[c(x_0'); d(x_0')]$ and that $y' \mapsto W(x_0', y')$ is bounded from below on $[c(x_0'); d(x_0')]$. Then :

$$\forall x' \in I_k^L, \lim_{n\to+\infty} \sup_{T_L^n(x)=x'} e^{A_R^n(S_R^n(x,d(x)))} = 0$$

PROOF : Let Γ_n be the set of the branches of T_L of length n arriving in I_k^L, such that :

$$\sup_{T_L^n(x)=x'} e^{A_R^n(S_R^n(x,d(x)))} = \sup_{\gamma\in\Gamma_n} e^{A_R^n(S_R^n(\gamma^{-1}(x'),d(\gamma^{-1}(x'))))}$$

Fix $\gamma \in \Gamma_n$. For every $x' \in I_k^L$ and $j \in [\![0\,;\,n]\!]$, $T_L^j(\gamma^{-1}(x'))$ always lies in the same interval $I_{\kappa_j(\gamma)}^L$ of the Markov partition of T_L. Thus $\gamma^{-1}(x')$ is always in the same T_L-cylinder of length n regardless of the choice of $x' \in I_k^L$. In particular, $d(\gamma^{-1}(x'))$ is independent of x', hence the same goes for $S_R^n(\gamma^{-1}(x'), d(\gamma^{-1}(x')))$ and for our supremum.

Thanks to the first point, it is enough to compute the limit for at single point x_0'. Extending the cocycle relation between A_L^n, A_R^n and W to $\tilde{C}$, we get that for every $x \in \mathbb{S}^1$ such that $T_L^n(x) = x'$:

$$A_L^n(x) + W(x, d(x)) = W(T_L^n(x), S_R^n(x, d(x))) + A_R^n(S_R^n(x, d(x)))$$

Hence :

$$e^{\Re A_R^n(S_R^n(x,d(x)))} = \frac{e^{\Re W(x,d(x))}}{e^{\Re W(x',S_R^n(x,d(x))}} e^{\Re A_L^n(x)} \leq \frac{\sup_{x\in\mathbb{S}^1} e^{\Re W(x,d(x))}}{\inf_{y'\in[c(x');d(x')]} e^{\Re W(x',y')}} \sup_{T_L^n(x)=x'} e^{\Re A_L^n(x)}$$

which goes to 0 when n goes to the infinity thanks to one of the previous lemmas and the uniform convergence of the series. ■

Lemma 2.30. *Suppose that ψ and $x \mapsto W(x, d(x))$ are bounded on $\mathbb{S}^1$, and that ψ is such that $x \mapsto \psi(x)e^{-W(x,d(x))}$ has bounded variations on the closure of each interval of the Markov partition of T_L. Let $x_0' \in I_k^L$ for some k such that the function series :*

$$y' \mapsto \sum_{n\geq 0} e^{\Re A_L^n(S_L^n(x_0',y'))}$$

converges uniformly on $[c(x_0'); d(x_0')] = \overline{J_k^R}$ and that $y' \mapsto W(x_0', y')$ is absolutely continuous on $\overline{J_k^R}$. Then for every $x_1' \in I_k^L$ for which the function series :

$$y' \mapsto \sum_{n\geq 0} e^{\Re A_L^n(S_L^n(x_1',y'))}$$

converge pointwise on $\overline{J_k^R}$ and $y' \mapsto W(x_1', y')$ is also absolutely continuous on $\overline{J_k^R}$, we have :

$$\forall y' \in \overline{J_k^R}, h(x_1', y') = h(x_0', y')$$

PROOF : The boundedness of ψ, the absolute continuity hypothesis on $y' \mapsto W(x'_i, y')$ and the pointwise convergence of the series :

$$\sum_{n\geq 0} e^{\Re A_L^n(S_L^n(x'_i,y'))}$$

for every $y' \in \overline{J_k^R}$ are enough to ensure the pointwise convergence of $(h_n(x'_i, y'))_{n\geq 0}$ to $h(x'_i, y')$ ($i = 0, 1$). We now reuse the notations of lemma 2.29. For $\gamma \in \Gamma_n$, $d(\gamma) = d(\gamma^{-1}(x'))$ where x' is any point of I_k^L is well defined, and we can write for $i = 0, 1$:

$$h_n(x'_i, y') = \sum_{\gamma\in\Gamma_n} \psi(\gamma^{-1}(x'_i))e^{A_L^n(\gamma^{-1}(x'_i))-W(T_C^n(\gamma^{-1}(x'_i),d(\gamma)))}\mathbb{1}_{]c(x'_i);y']}(S_R^n(\gamma^{-1}(x'_i), d(\gamma)))$$

But for every $(x, y) \in \tilde{C}$ we have :

$$A_L^n(x) + W(x, y) = W(T_C^n(x, y)) + A_R^n(S_R^n(x, y))$$

so this yields :

$$h_n(x'_i, y') = \sum_{\gamma\in\Gamma_n} \psi(\gamma^{-1}(x'_i))e^{A_R^n(S_R^n(\gamma^{-1}(x'_i),d(\gamma)))-W(\gamma^{-1}(x'_i),d(\gamma))}\mathbb{1}_{]c(x'_i);y']}(S_R^n(\gamma^{-1}(x'_i), d(\gamma)))$$

Noting that $S_R^n(\gamma^{-1}(x'_i), d(\gamma))$ only depends on $(\kappa_j(\gamma))_{j\in[\![0\,;\,n]\!]}$ and not on $x'_i \in I_k^L$, we get :

$$\begin{aligned}
&|h_n(x'_1, y') - h_n(x'_0, y')| \\
&\leq \sum_{\gamma\in\Gamma_n} \left|\psi(\gamma^{-1}(x'_1))e^{-W(\gamma^{-1}(x'_1),d(\gamma))} - \psi(\gamma^{-1}(x'_0))e^{-W(\gamma^{-1}(x'_0),d(\gamma))}\right| e^{\Re A_R^n(S_R^n(\gamma^{-1}(x'_0),d(\gamma)))} \\
&\leq \sup_{T_L^n(x)=x'_0} e^{\Re A_R^n(S_R^n(x,d(x)))} \sum_{\gamma\in\Gamma_n} \left|\psi(\gamma^{-1}(x'_1))e^{-W(\gamma^{-1}(x'_1),d(\gamma))} - \psi(\gamma^{-1}(x'_0))e^{-W(\gamma^{-1}(x'_0),d(\gamma))}\right|
\end{aligned}$$

where this last sum is bounded from above thanks to the bounded variations of the map $x \mapsto \psi(x)e^{-W(x,d(x))}$ (the intervals $([\gamma^{-1}(x'_0); \gamma^{-1}(x'_1)[)_{\gamma\in\Gamma_n}$ are pairwise disjoint) and the supremum goes to 0 whenever n goes to the infinity thanks to lemma 2.29 (the boundedness of $y' \mapsto W(x'_0, y')$ is implied by its absolute continuity). ■

Lemma 2.31. *For every $x' \in \mathbb{S}^1$, the restriction of the map :*

$$\begin{array}{rccl}
\delta_{x'} & : & J^R(x') & \rightarrow \mathbb{C} \\
 & & y' & \mapsto e^{-A_R(y')}h(x', y') - h(T_C^{-1}(x', y'))
\end{array}$$

to any interval I_k^R of the Markov partition of T_R included in the fiber $J^R(x')$ can be extended to an absolutely continuous function on $\overline{I_k^R}$. Moreover, we have that for Lebesgue almost every $y' \in J^R(x')$:

$$\delta'_{x'}(y') = \left[e^{-A_R}\right]'(y')h(x', y')$$

Note that $\delta_{x'}$ is not globally absolutely continuous on $J^R(x')$, as it does not need to be continuous at the partition points.

PROOF : Let $(x', y') \in C$, so that $\delta_{x'}(y')$ is well defined, and note $x = S_L(x', y')$. x does not depend on the choice of $y' \in I_k^R$, but only on x' and k. Substituting the expression of h :

$$h(x', y') = e^{-W(x',y')}g(x', y') - \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right]'(x', t)g(x', t)dt$$

inside the definition of δ, we get :

$$\begin{aligned}
\delta_{x'}(y') = {}& e^{-W(x',y')-A_R(y')}g(x', y') - e^{-A_R(y')} \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x', u)g(x', u)du \\
& - e^{-W(T_C^{-1}(x',y'))}g(T_C^{-1}(x', y')) + \int_{c(S_L(x',y'))}^{T_R(y')} \left[\partial_2 e^{-W}\right](S_L(x', y'), t)g(S_L(x', y'), t)dt
\end{aligned}$$

However, recall that on one hand :

$$g(x',y') = g(T_C^{-1}(x',y'))e^{A_L(S_L(x',y'))} + \eta(S_L(x',y'))$$

and on the other hand :

$$W(x',y') + A_R(y') = W(T_C^{-1}(x',y')) + A_L(S_L(x',y'))$$

This implies that the two non-integral terms can be simplified to :

$$\begin{aligned} e^{-W(x',y')-A_R(y')}g(x',y') &- e^{-W(T_C^{-1}(x',y'))}g(T_C^{-1}(x',y')) \\ &= \left(g(x',y')e^{-A_L(S_L(x',y'))} - g(T_C^{-1}(x',y'))\right) e^{-W(T_C^{-1}(x',y'))} \\ &= \eta(S_L(x',y'))e^{-A_L(S_L(x',y'))}e^{-W(T_C^{-1}(x',y'))} \\ &= \eta(x)e^{-W(x',y')-A_R(y')} \end{aligned}$$

Hence :

$$\begin{aligned} \delta_{x'}(y') = \eta(x)e^{-W(x',y')-A_R(y')} &- e^{-A_R(y')} \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x',u)g(x',u)du \\ &+ \int_{c(x)}^{T_R(y')} \left[\partial_2 e^{-W}\right](x,t)g(x,t)dt \end{aligned}$$

Now, since T_R is continuously differentiable on $\overline{I_k^R}$, that e^{-A_R} is absolutely continuous on $\overline{I_k^R}$, that $t \mapsto e^{-W(x,t)}$ is absolutely continuous on $\overline{J^R(x)}$, and that $t \mapsto g(x,t)$ is essentially bounded on $\overline{J^R(x)}$ (by continuity), we get that $y' \mapsto \delta_{x'}(y')$ extends to an absolutely continuous function on $\overline{I_k^R}$ with derivative Lebesgue almost everywhere equal to :

$$\begin{aligned} \delta'_{x'}(y') = \eta(x)\left[e^{-W(x',.)-A_R(.)}\right]'(y') &- \left[e^{-A_R}\right]'(y') \int_{c(x')}^{y'} \left[\partial_2 e^{-W}\right](x',u)g(x',u)du \\ - e^{-A_R(y')}\left[e^{-W(x',.)}\right]'(y')g(x',y') &+ B'(y') \end{aligned}$$

where :

$$B(y') = \int_{c(x)}^{T_R(y')} \left[\partial_2 e^{-W}\right](x,t)g(x,t)dt$$

From now on we will assume that y' is such that $\delta'_{x'}(y')$ exists.

Let us handle the term $B(y')$ separately. First observe that since $S_L(x',u)$ only depends on x' and on the interval of the T_R-partition in which u lies, we have $S_L(x',u) = x = S_L(x',y')$ for every $u \in I_k^R$. Now fix any $z' \in I_k^R$, and split the integral defining B at $T_R(z')$:

$$B(y') = \int_{c(x)}^{T_R(z')} \left[\partial_2 e^{-W}\right](x,t)g(x,t)dt + \int_{T_R(z')}^{T_R(y')} \left[\partial_2 e^{-W}\right](x,t)g(x,t)dt$$

The first term being constant, we only have to compute the derivative of the second one. Since $y', z' \in I_k^R$ and T_R is an orientation-preserving $\mathcal{C}^1$-diffeomorphism on I_k^R, the change of variable $t = T_R(u)$ yields :

$$\int_{T_R(z')}^{T_R(y')} \left[\partial_2 e^{-W}\right](x,t)g(x,t)dt = \int_{z'}^{y'} \left[\partial_2 e^{-W}\right](x,T_R(u))g(x,T_R(u))T_R'(u)du$$

By remarking that :

$$\left[\partial_2 e^{-W}\right](x,T_R(u))T_R'(u) = \left[e^{-W(x,T_R(.))}\right]'(u)$$

for Lebesgue almost every u, we end up with :

$$B'(y') = \left[e^{-W(x,T_R(.))}\right]'(y')g(x,T_R(y'))$$

But since $x = S_L(x', u)$ for any $u \in I_k^R$, we also have :

$$\left[e^{-W(x,T_R(.))}\right]'(y') = \left[e^{-W(S_L(x',.),T_R(.))}\right]'(y') = \left[e^{-W(T_C^{-1}(x',.))}\right]'(y')$$

Hence :

$$\begin{aligned} B'(y') &= \left[e^{-W(T_C^{-1}(x',.))}\right]'(y')g(T_C^{-1}(x',y')) \\ &= \left[e^{-W(T_C^{-1}(x',.))}\right]'(y')\left(g(x',y') - \eta(x)\right)e^{-A_L(x)} \\ &= \left[e^{-W(T_C^{-1}(x',.))-A_L(x)}\right]'(y')\left(g(x',y') - \eta(x)\right) \\ &= \left[e^{-W(x',.)-A_R(.)}\right]'(y')\left(g(x',y') - \eta(x)\right) \end{aligned}$$

We can now compute the final expression of $\delta'_{x'}(y')$:

$$\begin{aligned} \delta'_{x'}(y') = \eta(x)&\left[e^{-W(x',.)-A_R(.)}\right]'(y') - \left[e^{-A_R}\right]'(y')\int_{c(x')}^{y'}\left[\partial_2 e^{-W}\right](x',u)g(x',u)du \\ &- e^{-A_R(y')}\left[e^{-W(x',.)}\right]'(y')g(x',y') + \left[e^{-W(x',.)-A_R(.)}\right]'(y')\left(g(x',y') - \eta(x)\right) \\ = &- \left[e^{-A_R}\right]'(y')\int_{c(x')}^{y'}\left[\partial_2 e^{-W}\right](x',u)g(x',u)du \\ &+ \Bigg(- e^{-A_R(y')}\left[e^{-W(x',.)}\right]'(y') + \left[e^{-A_R}\right]'(y')e^{-W(x',y')} \\ &\quad + e^{-A_R(y')}\left[e^{-W(x',.)}\right]'(y')\Bigg)g(x',y') \\ = &\left[e^{-A_R}\right]'(y')\left(e^{-W(x',y')}g(x',y') - \int_{c(x')}^{y'}\left[\partial_2 e^{-W}\right](x',u)g(x',u)du\right) \\ = &\left[e^{-A_R}\right]'(y')h(x',y') \end{aligned}$$

■

Lemma 2.32. *Let $y_0 \in I_k^R$ for some $I_k^R \in \mathcal{P}^R$, and $x_1, x_2 \in \mathbb{S}^1$ such that $(x_1, y_0) \in C$ and $(x_2, y_0) \in C$. If there are $\lambda \in \mathbb{C}$ and an open neighbourhood U of y_0 in I_k^R such that :*

$$\forall y \in U, h(T_C^{-1}(x_1, y)) - h(T_C^{-1}(x_2, y)) = \lambda$$

then there is $\mu \in \mathbb{C}$ such that :

$$\forall y \in U, h(x_1, y) - h(x_2, y) = \mu$$

PROOF : Let $\varphi(y) = h(x_1, y) - h(x_2, y)$. Using the definition of δ, φ can be rewritten :

$$\begin{aligned} \forall y \in U, \varphi(y) &= \left(\delta_{x_1}(y) + h(T_C^{-1}(x_1, y))\right)e^{A_R(y)} - \left(\delta_{x_2}(y) + h(T_C^{-1}(x_2, y))\right)e^{A_R(y)} \\ &= (\delta_{x_1}(y) - \delta_{x_2}(y) + \lambda)e^{A_R(y)} \end{aligned}$$

According to lemma 2.31, φ is absolutely continuous on $U \subset I_k^R$ and its derivative satisfies for Lebesgue almost every $y \in U$:

$$\begin{aligned} \varphi'(y) &= \left(\left[e^{-A_R}\right]'(y)h(x_1, y) - \left[e^{-A_R}\right]'(y)h(x_2, y)\right)e^{A_R(y)} \\ &\quad + (\delta_{x_1}(y) - \delta_{x_2}(y) + \lambda)\left[e^{A_R}\right]'(y) \\ &= \varphi(y)\left[e^{-A_R}\right]'(y)e^{A_R(y)} + \varphi(y)e^{-A_R(y)}\left[e^{A_R}\right]'(y) \\ &= 0 \end{aligned}$$

which proves that φ is constant over U. ■

If $y \in \mathbb{S}^1$ and $n \geq 0$, let $U_n^L(y) =]S_L^n(a(y), y); S_L^n(b(y), y)[\subset J^L(T_R^n(y))$.

Lemma 2.33. *Let $y \in \mathbb{S}^1$ and $n \geq 0$. For every $x \in U_n^L(y) \cap \partial\mathcal{P}^L$:*

$$y \in J^R(T_L^n(x)) \cap [S_R^n(x, c(x)); S_R^n(x, d(x))[$$

PROOF : Pick $y \in \mathbb{S}^1$ and $n \geq 0$ such that $U_n^L(y) \cap \partial\mathcal{P}^L \neq \emptyset$. Take $x \in U_n^L(y) \cap \partial\mathcal{P}^L$, and let $w \in]a(y); b(y)[$ be such that $x = S_L^n(w, y)$. Observe that :

$$T_C^{-n}(w, y) = (S_L^n(w, y), T_R^n(y)) = (x, T_R^n(y))$$

with $T_R^n(y) \in [c(x); d(x)[$. Hence $w = \pi_1 T_C^n(x, T_R^n(y)) = T_L^n(x)$ and :

$$y = \pi_2 T_C^n(x, T_R^n(y)) = S_R^n(x, T_R^n(y)) \in [S_R^n(x, c(x)); S_R^n(x, d(x))[$$

This proves that $y \in J^R(T_L^n(x)) \cap [S_R^n(x, c(x)); S_R^n(x, d(x))[$. ■

Lemma 2.34. *For each pair of intervals I_j^L, I_k^L such that $J_j^R \cap J_k^R \neq \emptyset$, there is a constant $\lambda_{j,k} \in \mathbb{C}$ such that :*

$$\forall x_1 \in I_j^L, \forall x_2 \in I_k^L, \forall y \in J_j^R \cap J_k^R, h(x_1, y) - h(x_2, y) = \lambda_{j,k}$$

PROOF : First, note that if $j = k$, then $h(x_1, y) = h(x_2, y)$ for every $x_1, x_2 \in I_j^L = I_k^L$ and $y \in J_j^R = J_k^R$, i.e. $\lambda_{j,j} = 0$. Hence it is sufficient to prove the result for an unique pair of points $(x_1, x_2) \in I_j^L \times I_k^L$ for the indices $j \neq k$ such that $J_j^R \cap J_k^R \neq \emptyset$.

If $n \geq 0$, let :

$$W_n = \left(J_j^R \cap J_k^R\right) \setminus \bigcup_{x \in \partial\mathcal{P}^L} [S_R^n(x, c(x)); S_R^n(x, d(x))]$$

Those sets are increasing : for every $n \geq 0$, $W_n \subset W_{n+1}$. Indeed, take $y \in W_n$, and assume there is a $x \in \partial\mathcal{P}^L$ such that $y \in \left[S_R^{n+1}(x, c(x)); S_R^{n+1}(x, d(x))\right]$. We have :

$$\begin{aligned}\left[S_R^{n+1}(x, c(x)); S_R^{n+1}(x, d(x))\right] &= [S_R^n(T_L(x), S_L(x, c(x))); S_R^n(T_L(x), S_L(x, d(x)))] \\ &\subset [S_R^n(T_L(x), c(T_L(x))); S_R^n(T_L(x), d(T_L(x)))]\end{aligned}$$

since $S_L(x, c(x)), S_L(x, d(x)) \in [c(T_L(x)); d(T_L(x))]$. As $T_L(x) \in \partial\mathcal{P}^L$, this means that y is not in W_n, which is a contradiction.

We shall now prove that the reunion of all the W_n is $\left(J_j^R \cap J_k^R\right) \setminus \{y_1, \ldots, y_q\}$ where $q \leq N$. Suppose that there are $y_0, \ldots, y_N \in J_j^R \cap J_k^R$ that are in no W_n for any n. By the pigeonhole principle, there is for every $n \geq 0$ a $x_n \in \partial\mathcal{P}^L$ such that two different points y_{k_n}, y_{l_n} are both in $[S_R^n(x_n, c(x_n)); S_R^n(x_n, d(x_n))]$. Since the set of possible pairs of distinct points (y_k, y_l) is finite, one of them is seen infinitely many times, so there is a subsequence of integers $(n_m)_{m \geq 0}$ increasing to the infinity such that for every $m \geq 0$ $k_{n_m} = k$, $l_{n_m} = l$, $y_k \neq y_l$ and :

$$\forall m \geq 0, y_k, y_l \in [S_R^{n_m}(x_{n_m}, c(x_{n_m})); S_R^{n_m}(x_{n_m}, d(x_{n_m}))]$$

But the expansivity of T_R implies that the diameter of these intervals is converging to 0 whenever m goes to the infinity, so we must have $y_k = y_l$ which is a contradiction.

W_n is obtained by subtracting at most N intervals from an interval of $\mathbb{S}^1$, so it has a finite number of connex components bounded by $N + 1$ (each of them being actually an interval). We shall show that for every n and every connex component $Z_{j,n}$ of W_n, there is a $\lambda_{j,n} \in \mathbb{C}$ such that $h(x_1, y) - h(x_2, y) = \lambda_{j,n}$ for every $y \in Z_{j,n}$. We start by noting that the unit circle $\mathbb{S}^1$ is partitioned by the finite set of T_R-cylinders of length n, where each cylinder is an interval half-open to the right. Except finitely many points $y'_1, \ldots, y'_r$, every point of W_n is in the interior of one of those cylinders. The set $W_n \setminus \{y_1, \ldots, y_q, y'_1, \ldots y'_r\}$ is then the disjoint union of a finite number of open intervals, each of them being included in a T_R-cylinder of length n. Let V be such of an interval. For every $y \in V \subset W_n$, we have $U_n^L(y) \cap \partial\mathcal{P}^L = \emptyset$, which means that $S_L^n(x_1, y)$ and $S_L^n(x_2, y)$ are in the same interval of $\mathcal{P}^L$. This imples that :

$$\forall y \in V, h(T_C^{-n}(x_1, y)) = h(T_C^{-n}(x_2, y))$$

Moreover, $T_R^j(V)$ is for every $j \leq n$ an open neighbourhood of $T_R^j(y)$ that is included in the interior of an interval of the Markov partition for T_R (because V is included in the interior of a cylinder). Applying lemma 2.32 repeatedly for the neighbourhoods $T_R^j(V)$ of $T_R^j(y)$, we obtain the existence of a constant λ_V such that $h(x_1,y)-h(x_2,y)=\lambda_V$ for every $y\in V$. This proves that the map $y\in W_n\mapsto h(x_1,y)-h(x_2,y)$ is piecewise constant with a finite number of possible discontinuites. However, W_n is a subset of a fiber J_j^L on which this very same map is continuous, so it is actually constant on each connex component of W_n.

Finally, since the W_n are increasing, they overlap ; and each connex component of W_n must be included in a connex component of W_{n+1}. This ensures that $y\mapsto h(x_1,y)-h(x_2,y)$ is constant on each connex component of the reunion of all the W_n. But recall that this union is $\left(J_j^R\cap J_k^R\right)\setminus\{y_1,\dots,y_q\}$, hence the continuity of this map over the fibers implies that it is actually constant over $J_j^R\cap J_k^R$. ∎

With this result, we are now able to define the distribution associated with ψ. For every $1\le k\le N$, we write the interval $I_k^R\in\mathcal{P}^L$ as $I_k^R=[u_k;v_k[$, by supposing that these intervals are ordered in such a way that $v_k=u_{k+1 \mod N}$ for every k. We also pick a $x_k\in J_k^L$ for every k. Then let :

$$\begin{array}{rccc} h: & \mathbb{S}^1 & \to & \mathbb{C} \\ & y\in I_k^R & \mapsto & h_k+h(x_k,y) \end{array}$$

where $h_1=-h(x_1,u_1)$ and $h_{k+1}=h_k+h(x_k,v_k)-h(x_{k+1},u_{k+1})$ for every $k\le N-1$. Note that h_1 has been chosen so that $h(u_1)=0$.

This map h is clearly continuous on each I_k^R, but it is also continuous at each u_k when $2\le k\le N$. Indeed, if $u_k=v_{k+1}$, then :

$$h(u_k^+)-h(u_k^-)=h_k+h(x_k,u_k)-h_{k-1}-h(x_{k-1},u_k)=0$$

by definition of h_k in terms of h_{k-1}. Moreover, h does not depend on the choice of the x_k in the fibers over each interval : if $\tilde h$ is the function obtained for another choice of $\tilde x_1\in J_1^L,\dots,\tilde x_N\in J_N^L$, then for every $1\le k\le N$ and $y\in I_k^R$ we see that :

$$h(y)-\tilde h(y)=h_k+h(x_k,y)-\tilde h_k-h(\tilde x_k,y)=\mu_k$$

does not depend on $y\in I_k^R$. But since the map $y\mapsto h(y)-\tilde h(y)$ is continuous at each u_k between two intervals, we get that $\mu_k=\mu_{k+1}$ for any $1\le k\le N-1$, i.e. that $h-\tilde h$ is constantly equal to $h(u_1)-\tilde h(u_1)=0$.

Thanks to this map h, we can now define the distribution ν that will be our candidate for the preimage of ψ by Φ. Let $\tilde h:\mathbb{R}\to\mathbb{C}$ the continuous lift of h to $\mathbb{R}$. If $\varphi\in E_R$, we define :

$$\langle\nu,\varphi\rangle=\tilde h(2\pi)\tilde\varphi(2\pi)-\tilde h(0)\tilde\varphi(0)-\int_0^{2\pi}\tilde h(t)\tilde\varphi(t)dt$$

such that it can naturally be extended to act on any test function $\varphi\in\mathcal{C}^1\left(\left[e^{ia};e^{ib}\right[\right)$ by :

$$\langle\nu,\varphi\mathbb{1}_{\left[e^{ia};e^{ib}\right[}\rangle=\tilde h(b)\tilde\varphi(b)-\tilde h(a)\tilde\varphi(a)-\int_a^b\tilde h(t)\tilde\varphi(t)dt$$

Lemma 2.35. *Let $\varphi\in\mathcal{C}^1(\overline{I_k^R})$ for some k. Let $\gamma=T_{R/I_k^R}$. Then :*

$$\langle\nu,e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\mathbb{1}_{\gamma(I_k^R)}\rangle=\langle\nu,\varphi\mathbb{1}_{I_k^R}\rangle$$

PROOF : We note $I_k^R=[a;b[=\left[e^{i\tilde a};e^{i\tilde b}\right[$, such that :

$$\begin{aligned}\langle\nu,e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\mathbb{1}_{\gamma(I_k^R)}\rangle=&\left[e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\right](\gamma(b))h(\gamma(b))\\&-\left[e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\right](\gamma(a))h(\gamma(a))\\&-\int_{\tilde{\gamma(a)}}^{\tilde{\gamma(b)}}\left[e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\right]'(t)\tilde h(t)dt\end{aligned}$$

By means of a change of variable $u = \tilde{\gamma}(t)$ under this integral, we get :

$$\int_{\gamma(\tilde{a})}^{\gamma(\tilde{b})} \left[e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\right]'(t)\tilde{h}(t)dt = \int_{\tilde{a}}^{\tilde{b}} \left[e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\right]'(\tilde{\gamma}(u))\tilde{h}(\tilde{\gamma}(u))|\tilde{\gamma}'(u)|du$$

But by noting that for Lebesgue almost every $u \in \left[\tilde{a};\tilde{b}\right[$:

$$\left[e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\right]'(\tilde{\gamma}(u))|\tilde{\gamma}'(u)| = \left[e^{A_R\circ\gamma^{-1}\circ\gamma}\varphi\circ\gamma^{-1}\circ\gamma\right]'(u) = \left[e^{A_R}\varphi\right]'(u)$$

this gives :

$$\begin{aligned}\langle \nu, e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\mathbb{1}_{\gamma(I_k^R)}\rangle = {} & e^{A_R(b)}\varphi(b)h(\gamma(b)) - e^{A_R(a)}\varphi(a)h(\gamma(a)) \\ & - \int_{\tilde{a}}^{\tilde{b}} \left[e^{A_R}\varphi\right]'(u)\tilde{h}(\tilde{\gamma}(u))du\end{aligned}$$

Comparing with :

$$\langle \nu, \varphi\mathbb{1}_{I_k^R}\rangle = \varphi(b)h(b) - \varphi(a)h(a) - \int_{\tilde{a}}^{\tilde{b}} \tilde{\varphi}'(u)\tilde{h}(u)du$$

we get that the difference of both expressions is :

$$\begin{aligned}\langle \nu, e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\mathbb{1}_{\gamma(I_k^R)}\rangle - \langle \nu, \varphi\mathbb{1}_{I_k^R}\rangle = {} & \varphi(b)d(b) - \varphi(a)d(a) \\ & - \int_{\tilde{a}}^{\tilde{b}} \tilde{\varphi}'(t)\tilde{d}(t)dt - \int_{\tilde{a}}^{\tilde{b}} \left[e^{\tilde{A_R}}\right]'(t)\tilde{\varphi}(t)\tilde{h}(\tilde{\gamma}(t))dt\end{aligned}$$

where :

$$\forall y \in I_k^R, d(y) = e^{A_R(y)}h(\gamma(y)) - h(y) = e^{A_R(y)}h(T_R(y)) - h(y)$$

Let $y \in I_k^R$, such that $T_R(y) \in I_k^L$. We have :

$$d(y) = e^{A_R(y)}(h_l + h(x_l, T_R(y))) - (h_k + h(x_k, y))$$

for some $x_k \in J_k^L$ and $x_l \in J_l^L$. Since the definition of h does not depend on the choice of the x_i, we can assume that $x_l = S_L(x_k, y) \in J_l^L$, which gives us that :

$$\begin{aligned} d(y) &= e^{A_R(y)}(h_l + h(S_L(x_k,y), T_R(y))) - (h_k + h(x_k, y)) \\ &= e^{A_R(y)}h_l - h_k + e^{A_R(y)}\left(h(T_C^{-1}(x_k,y)) - e^{-A_R(y)}h(x_k,y)\right) \\ &= e^{A_R(y)}h_l - h_k - e^{A_R(y)}\delta_{x_k}(y)\end{aligned}$$

where both $y \mapsto e^{A_R(y)}$ and $y \mapsto \delta_{x_k}(y)$ extend to absolutely continuous maps on $\overline{I_k^L}$. Hence d extends to an absolutely continuous map on $\overline{I_k^L}$, and thanks to lemma 2.31 we get that its derivative is for Lebesgue almost every y :

$$\begin{aligned} d'(y) &= \left[e^{A_R}\right]'(y)h_l - \left[e^{A_R}\right]'(y)\delta_{x_k}(y) - e^{A_R(y)}\delta'_{x_k}(y) \\ &= -\left[e^{A_R}\right]'(y)\left(-h_l + e^{-A_R(y)}h(x_k,y) - h(T_C^{-1}(x_k,y))\right) - e^{A_R(y)}\left[e^{-A_R}\right]'(y)h(x_k,y) \\ &= -\left[e^{A_R-A_R}\right]'(y)h(x_k,y) + \left[e^{A_R}\right]'(y)(h_l + h(T_C^{-1}(x_k,y))) \\ &= \left[e^{A_R}\right]'(y)(h_l + h(x_l, T_R(y))) \\ &= \left[e^{A_R}\right]'(y)h(T_R(y))\end{aligned}$$

This yields :

$$
\begin{aligned}
-\int_{\tilde{a}}^{\tilde{b}} \tilde{\varphi}'(t)\tilde{d}(t)dt - \int_{\tilde{a}}^{\tilde{b}} \left[e^{\tilde{A_R}}\right]'(t)\tilde{\varphi}(t)\tilde{h}(\tilde{\gamma}(t))dt &= -\int_{\tilde{a}}^{\tilde{b}} \tilde{\varphi}'(t)\tilde{d}(t)dt - \int_{\tilde{a}}^{\tilde{b}} \tilde{\varphi}(t)\tilde{d}'(t)dt \\
&= -\int_{\tilde{a}}^{\tilde{b}} \left[\tilde{\varphi}\tilde{d}\right]'(t)dt \\
&= \tilde{\varphi}(\tilde{a})\tilde{d}(\tilde{a}) - \tilde{\varphi}(\tilde{b})\tilde{d}(\tilde{b})
\end{aligned}
$$

which proves that $\langle \nu, e^{A_R\circ\gamma^{-1}}\varphi\circ\gamma^{-1}\mathbb{1}_{\gamma(I_k^R)}\rangle = \langle \nu, \varphi\mathbb{1}_{I_k^R}\rangle$. ■

It is now easy to prove that ν is indeed an eigendistribution of the Ruelle operator of T_R for the eigenvalue 1.

Lemma 2.36. *For every $\varphi \in E_R$:*

$$\langle \nu, \mathcal{L}_R\varphi\rangle = \langle \nu, \varphi\rangle$$

PROOF : Any $\varphi \in E_R$ can be written :

$$\varphi = \sum_{k=1}^{N} \varphi_k \mathbb{1}_{I_k^R}$$

where each $\varphi_k = \varphi_{/I_k^R} \in \mathcal{C}^1(\overline{I_k^R})$. Moreover, for every $y \in \mathbb{S}^1$:

$$\mathcal{L}_R\left[\varphi_k\mathbb{1}_{I_k^R}\right](y) = \sum_{T_R(y')=y} e^{A_R(y')}\varphi_k(y')\mathbb{1}_{I_k^R}(y') = e^{A_R\circ\gamma_k^{-1}(y)}\varphi_k\circ\gamma_k^{-1}(y)\mathbb{1}_{\gamma_k(I_k^R)}(y)$$

where $\gamma_k = T_{R/I_k^R}$, as y may have only at most one preimage $\gamma_k^{-1}(y)$ in I_k^R. By linearity of ν and thanks to lemma 2.35, we get :

$$
\begin{aligned}
\langle \nu, \mathcal{L}_R\varphi\rangle = \sum_{k=1}^{N}\langle \nu, \mathcal{L}_R\left[\varphi_k\mathbb{1}_{I_k^R}\right]\rangle &= \sum_{k=1}^{N}\langle \nu, e^{A_R\circ\gamma_k^{-1}}\varphi_k\circ\gamma_k^{-1}\mathbb{1}_{\gamma_k(I_k^R)}\rangle \\
&= \sum_{k=1}^{N}\langle \nu, \varphi_k\mathbb{1}_{I_k^R}\rangle = \langle \nu, \varphi\rangle
\end{aligned}
$$ ■

Finally, we have to check that this ν is actually a preimage of ψ by Φ.

Lemma 2.37. $\psi = \Phi(\nu)$.

PROOF : Let $x \in \mathbb{S}^1$. The fiber $J^R(x) = [c(x); d(x)[$ has been assumed to be an union of intervals of the Markov partition of T_R, so possibly at the cost of reordering said intervals we can suppose that there are integers $p \leq q$ such that :

$$[c(x); d(x)[= \bigsqcup_{k=p}^{q} I_k^R$$

We note $I_k^R = [u_k; v_k[$, so that $c(x) = u_p$, $d(x) = v_q$ and $v_k = u_{k+1}$ for any $k \in [\![p\,;\, q-1]\!]$. We also recall that if $y \in I_k^R$, then $h(y) = h_k + h(x_i, y)$ where x_i can be chosen arbitrarily in J_k^L. Since $x \in J^L(y)$ for every $y \in [c(x); d(x)[$, we can choose to pick $x_k = x$ for every $k \in [p; q]$.

From there, we get that :

$$\begin{aligned}\Phi(\nu)(x) &= \langle \nu, e^{W(x,.)}\mathbb{1}_C(x,.)\rangle \\ &= e^{W(x,d(x))}h(d(x)) - e^{W(x,c(x))}h(c(x)) - \int_{c(x)}^{d(x)} \left[\partial_2 e^W\right](x,t)h(t)dt \\ &= e^{W(x,v_q)}(h_q + h(x,v_q)) - e^{W(x,u_p)}(h_p + h(x,u_p)) \\ &\qquad - \sum_{k=p}^{q} \int_{u_k}^{v_k} \left[\partial_2 e^W\right](x,t)(h_k + h(x,t))dt\end{aligned}$$

However, note that for every $k \in [p; q-1]$ we have :

$$h_{k+1} = h_k + h(x, v_k) - h(x, u_{k+1}) = h_k$$

so that $h_p = h_{p+1} = \ldots = h_{q-1} = h_q$, which yields :

$$\begin{aligned}\Phi(\nu)(x) &= \left(e^{W(x,v_q)} - e^{W(x,u_p)} - \int_{u_p}^{v_q} \left[\partial_2 e^W\right](x,t)dt\right) h_p \\ &\quad + \left(e^{W(x,v_q)}h(x,v_q) - e^{W(x,u_p)}h(x,u_p) - \int_{u_p}^{v_q} \left[\partial_2 e^W\right](x,t)h(x,t)dt\right) \\ &= e^{W(x,d(x))}h(x,d(x)) - e^{W(x,c(x))}h(x,c(x)) - \int_{c(x)}^{d(x)} \left[\partial_2 e^W\right](x,t)h(x,t)dt\end{aligned}$$

By lemma 2.27, this imples that $\Phi(\nu)(x) = \psi(x)$. ■

3. Appendix

3.1. Technical proofs.

Lemma 3.1. *Let I be a non-empty half-open interval of $\mathbb{S}^1$, $\gamma : \overline{I} \to \gamma(\overline{I})$ a $\mathcal{C}^1$-diffeomorphism, $h : \mathbb{S}^1 \to \mathbb{C}$ a continuous map over $\overline{I} \cup \gamma(\overline{I})$, and $f : I \to \mathbb{C}$ an absolutely continuous map over $\overline{I}$. Let ν be the linear operator given by the weak derivative of h. If for every $\varphi : I \to \mathbb{C}$ absolutely continuous on $\overline{I}$:*

$$\left\langle \nu, (\varphi f) \circ \gamma^{-1}\mathbb{1}_{\gamma(I)}\right\rangle = \langle \nu, \varphi \mathbb{1}_I\rangle$$

then for every half-open sub-inteval J of I and every $\varphi : J \to \mathbb{C}$ absolutely continuous on $\overline{J}$:

$$\left\langle \nu, (\varphi f) \circ \gamma^{-1}\mathbb{1}_{\gamma(J)}\right\rangle = \langle \nu, \varphi \mathbb{1}_J\rangle$$

Proof : Note $I = [a; b[$. For every $c, d \in \overline{I}$, $c < d$ and every $\varphi : [c; d[\to \mathbb{S}^1$ absolutely continuous on $[c; d]$, we have on one hand :

$$\langle \nu, \varphi \mathbb{1}_{[c;d[}\rangle = \varphi(d)h(d) - \varphi(c)h(c) - \int_c^d \varphi'(x)h(x)dx$$

and on the other hand :

$$\begin{aligned}\left\langle \nu, (\varphi f) \circ \gamma^{-1}\mathbb{1}_{\gamma[c;d[}\right\rangle &= \varphi(d)f(d)h(\gamma(d)) - \varphi(c)f(c)h(\gamma(c)) \\ &\qquad - \int_{\gamma(c)}^{\gamma(d)} \left[(\varphi f) \circ \gamma^{-1}\right]'(x)h(x)dx\end{aligned}$$

where this last integral can be rewritten as :

$$\begin{aligned}\int_{\gamma(c)}^{\gamma(d)} \left[(\varphi f)\circ\gamma^{-1}\right]'(x)h(x)dx &= \int_c^d \gamma'(x)\left[(\varphi f)\circ\gamma^{-1}\right]'(\gamma(x))h(\gamma(x))dx\\ &= \int_c^d [\varphi f]'(x)h(\gamma(x))dx\\ &= \int_c^d \varphi'(x)f(x)h(\gamma(x))dx\\ &\quad + \int_c^d \varphi(x)f'(x)h(\gamma(x))dx\end{aligned}$$

Let $\delta(x) = f(x)h(\gamma(x)) - h(x)$ and $\lambda(x) = f'(x)h(\gamma(x))$. Subtracting those two relations, we can write :

$$\begin{aligned}&\left\langle \nu, (\varphi f)\circ\gamma^{-1}\mathbb{1}_{\gamma[c;d[}\right\rangle - \left\langle \nu, \varphi\mathbb{1}_{[c;d[}\right\rangle\\ &\qquad = \varphi(d)\delta(d) - \varphi(c)\delta(c) - \int_c^d \varphi'(x)\delta(x)dx - \int_c^d \varphi(x)\lambda(x)dx\end{aligned}$$

We will now prove that δ is absolutely continuous over $\overline{I}$. For $c = a$ and $d = b$, one has :

$$\varphi(b)\delta(b) - \varphi(a)\delta(a) - \int_a^b \varphi'(x)\delta(x)dx = \int_a^b \varphi(x)\lambda(x)dx$$

Pick $y \in \,]a;b[$ and $\varepsilon > 0$. One can construct an increasing function $\varphi_\varepsilon \in \mathcal{C}^1(\overline{I})$ such that $\varphi_\varepsilon = 0$ on $[a; y-\varepsilon]$ and $\varphi_\varepsilon = 1$ on $[y;b]$. If we specialize the previous relation for this φ_ε, we get for ε sufficiently small that :

$$\delta(b) - \int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)\delta(x)dx = \int_a^b \varphi_\varepsilon(x)\lambda(x)dx \tag{1}$$

For every $x \in [a;b]\,, \lim\limits_{\varepsilon\to 0}\varphi_\varepsilon(x) = \mathbb{1}_{[y;b]}(x)$, hence by Lebesgue's dominated convergence theorem, we get that :

$$\lim_{\varepsilon\to 0}\int_a^b \varphi_\varepsilon(x)\lambda(x)dx = \int_y^b \lambda(x)dx$$

On the other hand, as $\displaystyle\int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)dx = \varphi_\varepsilon(y) - \varphi_\varepsilon(y-\varepsilon) = 1$ and $\varphi_\varepsilon'(x) \geq 0$:

$$\begin{aligned}\left|\int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)\delta(x)dx - \delta(y)\right| &= \left|\int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)(\delta(x)-\delta(y))dx\right|\\ &\leq \int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)|\delta(x)-\delta(y)|dx\end{aligned}$$

Since h is continuous over $\overline{I}\cup\gamma(\overline{I})$ and f is continuous over $\overline{I}$, δ is also continuous over $\overline{I}$. This ensures that if $\alpha > 0$, there is $\varepsilon_0 > 0$ such that for every x that satisfies $|x-y| < \varepsilon_0$, $|\delta(x)-\delta(y)| < \alpha$. Now for any $\varepsilon < \varepsilon_0$:

$$\left|\int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)\delta(x)dx - \delta(y)\right| \leq \alpha\int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)dx = \alpha$$

Hence $\displaystyle\lim_{\varepsilon\to 0}\int_{y-\varepsilon}^{y} \varphi_\varepsilon'(x)\delta(x)dx = \delta(y)$. Taking the limit in 1, one finally gets that :

$$\delta(b) - \delta(y) = \int_y^b \lambda(x)dx$$

This shows that δ is absolutely continuous over $\overline{I}$ and that its derivative is Lebesgue almost everywhere equal to λ.

Now we get back to our previous computation for $c < d$:

$$\begin{aligned}\left\langle \nu, (\varphi f)\circ\gamma^{-1}\mathbb{1}_{\gamma[c;d[}\right\rangle - \langle \nu, \varphi\mathbb{1}_{[c;d[}\rangle \\ &= \varphi(d)\delta(d) - \varphi(c)\delta(c) - \int_c^d \varphi'(x)\delta(x)dx - \int_c^d \varphi(x)\delta'(x)dx \\ &= \varphi(d)\delta(d) - \varphi(c)\delta(c) - \int_c^d [\varphi\delta]'(x)dx = 0\end{aligned}$$

which proves the lemma. ■

Lemma 3.2. *Let $y \in [a;b]$, $c \in]a;b]$ and $\varphi : [a;b] \to \mathbb{C}$ absolutely continuous. Then :*

$$\varphi(y)\mathbb{1}_{]a;y]}(c) - \int_a^y \varphi'(t)\mathbb{1}_{]a;t]}(c)dt = \varphi(c)\mathbb{1}_{]a;y]}(c)$$

PROOF : First note that for every $t \in]a;b]$:

$$\mathbb{1}_{]a;t]}(c) = \mathbb{1}_{[c;b]}(t)$$

From there it follows that :

$$\begin{aligned}\int_a^y \varphi'(t)\mathbb{1}_{]a;t]}(c)dt &= \int_a^y \varphi'(t)\mathbb{1}_{[c;b]}(t)dt \\ &= \begin{cases} 0 & \text{if } y \in [a;c[\\ \int_c^y \varphi'(t)dt = \varphi(y)-\varphi(c) & \text{if } y \in [c;b] \end{cases} \\ &= (\varphi(y)-\varphi(c))\,\mathbb{1}_{[c;b]}(y) = (\varphi(y)-\varphi(c))\,\mathbb{1}_{]a;y]}(c)\end{aligned}$$

which proves the result. ■

Lemma 3.3. *Let $f, g : I \to \mathbb{C}$ continuous maps over $\overline{I}$ and $\varphi : I \to \mathbb{C}$ absolutely continuous over $\overline{I}$ such that :*

$$\forall y, y' \in I, g(y') - g(y) = \varphi(y')f(y') - \varphi(y)f(y) - \int_y^{y'} \varphi'(t)f(t)dt$$

If φ never vanishes on $\overline{I}$, then :

$$\forall y, y' \in I, f(y') - f(y) = \frac{g(y')}{\varphi(y')} - \frac{g(y)}{\varphi(y)} - \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)g(t)dt$$

PROOF : First note that if φ is absolutely continuous and does not vanish on $\overline{I}$, then $\frac{1}{\varphi}$ is also absolutely continuous on $\overline{I}$.

Fix $y, y' \in I$, and expand :

$$\begin{aligned}\delta(y', y) &= \frac{g(y')}{\varphi(y')} - \frac{g(y)}{\varphi(y)} - \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)g(t)dt \\ &= \frac{1}{\varphi(y')}\left(g(y) + \varphi(y')f(y') - \varphi(y)f(y) - \int_y^{y'} \varphi'(t)f(t)dt\right) \\ &\quad - \frac{g(y)}{\varphi(y)} - \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)\left(g(y) + \varphi(t)f(t) - \varphi(y)f(y) - \int_y^t \varphi'(u)f(u)du\right)dt.\end{aligned}$$

Let $\lambda(t) = \displaystyle\int_y^t \varphi'(u)f(u)du$, which is an absolutely continuous map over $\overline{I}$ whose derivative is Lebesgue almost everywhere equal to $\varphi' f$. We get that :

$$\begin{aligned}\delta(y',y) &= f(y') + g(y)\underbrace{\left(\frac{1}{\varphi(y')} - \frac{1}{\varphi(y)} - \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right) dt\right)}_{=0} \\ &\quad + f(y)\varphi(y)\underbrace{\left(\int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right) dt - \frac{1}{\varphi(y')}\right)}_{=-\frac{1}{\varphi(y)}} \\ &\quad - \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)\varphi(t)f(t)dt - \frac{\lambda(y')}{\varphi(y')} + \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)\lambda(t)dt.\end{aligned}$$

By integrating by parts (recall that $\lambda(y) = 0$) the last term can be rewritten :

$$\int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)\lambda(t)dt = \frac{\lambda(y')}{\varphi(y')} - \frac{\lambda(y)}{\varphi(y)} - \int_y^{y'} \frac{\lambda'(t)}{\varphi(t)}dt = \frac{\lambda(y')}{\varphi(y')} - \int_y^{y'} \frac{1}{\varphi(t)}\varphi'(t)f(t)dt$$

Then :

$$\begin{aligned}\delta(y',y) &= f(y') - f(y) - \int_y^{y'} \frac{\partial}{\partial t}\left(\frac{1}{\varphi(t)}\right)\varphi(t)f(t)dt - \int_y^{y'} \frac{1}{\varphi(t)}\varphi'(t)f(t)dt \\ &= f(y') - f(y) - \int_y^{y'} \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{\varphi(t)}\varphi(t)\right)}_{=1} f(t)dt \\ &= f(y') - f(y)\end{aligned}$$

■

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