PERIODIC SOLUTIONS OF DISCONTINUOUS SECOND ORDER DIFFERENTIAL SYSTEMS

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Abstract. We provide sufficient conditions por the existence of periodic solutions of some classes of autonomous and non–autonomous second order differential equations with discontinuous right–hand sides. In the plane the discontinuities considered are given by the straight lines either $x = 0$, or $xy = 0$. Two applications of these results are made, one to control systems with variable structure, and the other to small external periodic excitation of a discontinuous nonlinear oscillator.

1. Introduction and statement of the main result

In these last tens the study of discontinuous differential systems became relevant in the boundary between Mathematics, Physics and Engineering. In the book [2] and in the survey [10] there are different models coming from the impacting motion in mechanical systems, or from switchings in electronic systems, or from hybrid dynamics in control systems, and so on. All of these models are formulated with differential equations with discontinuous right–hand sides. Also, many studies have been done in the qualitative aspects of the phase space of discontinuous differential systems, see for instance the hundreds of references quoted in [2] and [10]; and also in the periodic orbits of these discontinuous systems, see for instance [11].

In this paper we are mainly interested in the study of the periodic solutions of autonomous and non–autonomous second order differential equations with discontinuous right–hand sides. Recently discontinuous second order differential equations have been studied for several authors, mainly non–autonomous ones. Thus, discontinuous differential equations of the form $u'' + u + \alpha\,\mathrm{sign}(y) = F(\theta)$ where F is a periodic function has been studied in [7]. In [5] periodic solutions of discontinuous differential equations of the form $u'' + G(u) = F(\theta)$ are analyzed, where F is periodic and continuous, and G is continuous except at $u = 0$. In [6] the authors studied the periodic solutions of the discontinuous differential equations $u'' + \eta\mathrm{sign}(u) = \alpha\sin(\beta t)$.

Our main results will provide sufficient conditions for the existence of periodic solutions of the following two classes of autonomous second order differential equations with discontinuous right–hand sides:

$$u'' + u + \varepsilon\alpha\,\mathrm{sign}(u)G(u, u') = \varepsilon H(u, u'), \tag{1}$$

$$u'' + u + \varepsilon\alpha\,\mathrm{sign}(uu')G(u, u') = \varepsilon H(u, u'). \tag{2}$$

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Here $u = u(t)$, $\alpha \in \mathbb{R}$ is a parameter, ε is a small parameter, G and H are C^1 functions, and the prime denotes derivative with respect to the variable t. Note that the differential equation (1) is discontinuous at $u = 0$, and that the differential equation (2) is discontinuous at $uu' = 0$.

We also shall provide sufficient conditions for the existence of periodic solutions of the following two classes of non–autonomous second order differential equations with discontinuous right–hand sides:

$$r'' + \varepsilon^2 \alpha \operatorname{sign}(\cos\theta) G(\theta, r, r') = \varepsilon^2 H(\theta, r, r'), \tag{3}$$

$$r'' + \varepsilon^2 \alpha \operatorname{sign}(\sin(2\theta)) G(\theta, r, r') = \varepsilon^2 H(\theta, r, r'). \tag{4}$$

Here (r, θ) are the polar coordinates of the plane, i.e. $x = r\cos\theta$ and $y = r\sin\theta$, $\alpha \in \mathbb{R}$ is a parameter, ε is a small parameter, G and H are C^1 functions in the variables r and r', the functions G and H are continuous and periodic in the variable θ of period 2π, and the prime denotes derivative with respect to the variable θ. Note that the differential equation (3) is discontinuous at the straight line $x = 0$ of the plane in cartesian coordinates, and that the differential equation (4) is discontinuous at the straight lines $xy = 0$.

Denoting $x = u$ and $y = u'$ the autonomous differential equations of second order (1) and (2), respectively can be written as the following differential systems of first order in the plane

$$\begin{aligned} \frac{dx}{dt} &= x' = y, \\ \frac{dy}{dt} &= y' = -x - \varepsilon\alpha \operatorname{sign}(x) G(x, y) + \varepsilon H(x, y); \end{aligned} \tag{5}$$

with the discontinuity set $x = 0$, and

$$\begin{aligned} \frac{dx}{dt} &= x' = y, \\ \frac{dy}{dt} &= y' = -x - \varepsilon\alpha \operatorname{sign}(xy) G(x, y) + \varepsilon H(x, y); \end{aligned} \tag{6}$$

with the discontinuity set $xy = 0$.

Denoting $x = r$ and $y = r'/\varepsilon$ the non–autonomous differential equations of second order (3) and (4), respectively can be written as the following differential systems of first order in the plane

$$\begin{aligned} \frac{dx}{d\theta} &= x' = \varepsilon y, \\ \frac{dy}{d\theta} &= y' = -\varepsilon\alpha \operatorname{sign}(x) G(\theta, x, y) + \varepsilon H(\theta, x, y); \end{aligned} \tag{7}$$

with the discontinuity set $x = 0$, and

$$\begin{aligned} \frac{dx}{d\theta} &= x' = \varepsilon y, \\ \frac{dy}{d\theta} &= y' = -\varepsilon\alpha \operatorname{sign}(xy) G(\theta, x, y) + \varepsilon H(\theta, x, y); \end{aligned} \tag{8}$$

with the discontinuity set $xy = 0$.

The following propositions provide sufficient conditions for the existence of periodic solutions for the discontinuous differential systems (5), (6), (7) and (8), respectively.

Proposition 1. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (5) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ for each simple zero r^* of the function*

$$
\begin{aligned}
f_1(r) = & \int_0^{2\pi} H(r\cos\theta, r\sin\theta)\sin\theta\, d\theta \\
& +\alpha\left(\int_{\pi/2}^{3\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta - \int_{-\pi/2}^{\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta\right),
\end{aligned}
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$.

Proposition 2. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (6) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ for each simple zero r^* of the function*

$$
\begin{aligned}
f_2(r) = & \int_0^{2\pi} H(r\cos\theta, r\sin\theta)\sin\theta\, d\theta \\
& -\alpha\left(\int_0^{\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta + \int_{\pi}^{3\pi/2} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta\right) \\
& +\alpha\left(\int_{\pi/2}^{\pi} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta + \int_{3\pi/2}^{2\pi} G(r\cos\theta, r\sin\theta)\sin\theta\, d\theta\right),
\end{aligned}
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$.

Proposition 3. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (7) has a periodic solution $(x(\theta,\varepsilon), y(\theta,\varepsilon))$ for each simple zero x^* of the function*

$$
f_3(x) = \int_0^{2\pi} H(\theta, x, 0)\, d\theta + \alpha\left(\int_{\pi/2}^{3\pi/2} G(\theta, x, 0)\, d\theta - \int_{-\pi/2}^{\pi/2} G(\theta, x, 0)\, d\theta\right),
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*, 0)$ when $\varepsilon \to 0$.

Proposition 4. *For $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (8) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ for each simple zero x^* of the function*

$$
\begin{aligned}
f_4(x) = & \int_0^{2\pi} H(\theta, x, 0)\, d\theta - \alpha\left(\int_0^{\pi/2} G(\theta, x, 0)\, d\theta + \int_{\pi}^{3\pi/2} G(\theta, x, 0)\, d\theta\right) \\
& +\alpha\left(\int_{\pi/2}^{\pi} G(\theta, x, 0)\, d\theta + \int_{3\pi/2}^{2\pi} G(\theta, x, 0)\, d\theta\right),
\end{aligned}
$$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*, 0)$ when $\varepsilon \to 0$.

The proof of these four propositions is given in section 2. The proofs are based in a recent result on the averaging theory applied to discontinuous differential systems obtained by the authors and also by Douglas Novaes, see the appendix.

In the study of control systems with variable structure appear the autonomous discontinuous second order differential equations similar to

$$
u'' + u + \varepsilon\alpha\, \mathrm{sign}(u) u u' = \varepsilon\frac{\alpha}{\pi}u', \tag{9}
$$

see for instance the book [1].

Corollary 5. *For $\varepsilon \neq 0$ sufficiently small the control system with variable structure (9) has one periodic solution $u(t,\varepsilon)$, such that $\sqrt{u(0,\varepsilon)^2 + u'(0,\varepsilon)^2} \to 3/4$ when $\varepsilon \to 0$.*

In the next corollary we apply Proposition 3 for studying the periodic solutions of the following small external periodic excitation of a discontinuous nonlinear oscillator

$$r'' + \varepsilon^2 \alpha \operatorname{sign}(\cos\theta)\left((2-3r)\cos\frac{\theta}{2}\right) = -\varepsilon^2 \frac{\sqrt{2}\,\alpha}{\pi}\, r^2. \tag{10}$$

Such kind of differential equations are considered in the book [12]. Note that equation (10) is a non–autonomous discontinuous second order differential equation.

Corollary 6. *For $\varepsilon \neq 0$ sufficiently small the small external periodic excitation of the discontinuous nonlinear oscillator* (10) *has two periodic solutions $r_k(\theta,\varepsilon)$ for $k = 1, 2$, such that $r_1(0,\varepsilon) \to \cos\theta$ and $r_2(0,\varepsilon) \to 2\cos\theta$ when $\varepsilon \to 0$.*

The proof of the two corollaries are given in section 3.

2. Proof of the propositions

In this section we prove the four propositions using the averaging theory for discontinuous differential systems described in the appendix.

Proof of Proposition 1. We write the discontinuous differential system (5) in polar coordinates (r,θ) where $x = r\cos\theta$ and $y = r\sin\theta$, and we obtain

$$\begin{aligned}
\frac{dr}{dt} &= \varepsilon\big(H(r\cos\theta, r\sin\theta) - \alpha \operatorname{sgn}(\cos\theta) G(r\cos\theta, r\sin\theta)\big)\sin\theta,\\
\frac{d\theta}{dt} &= -1 + \frac{\varepsilon}{r}\Big(\big(H(r\cos\theta, r\sin\theta) - \alpha \operatorname{sgn}(\cos\theta) G(r\cos\theta, r\sin\theta)\big)\cos\theta\Big).
\end{aligned}$$

Now taking as new independent variable the angle θ this previous discontinuous differential system becomes

$$\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon\big(\alpha \operatorname{sgn}(\cos\theta) G(r\cos\theta, r\sin\theta) - H(r\cos\theta, r\sin\theta)\big)\sin\theta + O(\varepsilon^2)\\
&= \varepsilon F(\theta, r) + O(\varepsilon^2).
\end{aligned} \tag{11}$$

This system is under the assumptions of Theorem 7, where the variables of this theorem are in our case $t = \theta$, $T = 2\pi$, $\mathbf{x} = r$, $\mathcal{M} = h^{-1}(0) = \{x = 0\}$. So we apply this theorem to our previous discontinuous differential equation and we compute

$$f(r) = \int_0^{2\pi} F(\theta, r)d\theta = f_1(r),$$

where $f_1(r)$ is the function defined in the statement of Proposition 1. Since by assumptions G and H are C^1 functions in their two variables, it follows that $f_1(r)$ is C^1. Consequently, if r^* is a simple zero of $f_1(r)$, i.e. $f_1(r^*) = 0$ and

$$\left.\frac{df_1}{dr}\right|_{r=r^*} \neq 0,$$

then the Brouwer degree $d_B(f_1, V, r^*) \neq 0$ being V a convenient open neighborhood of r^*, see for more details on the Brouwer degree [3] and [9]. Hence, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (11) has a periodic solution $r(\theta,\varepsilon)$ such that $r(0,\varepsilon) \to r^*$ when $\varepsilon \to 0$. Going back through the polar change of variables we get that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (5) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$. So, the proposition is proved. $\square$

Proof of Proposition 2. The discontinuous differential system (6) in polar coordinates (r, θ) becomes

$$\begin{aligned}\frac{dr}{dt} &= \varepsilon\big(H(r\cos\theta, r\sin\theta) - \alpha\,\mathrm{sgn}(\sin(2\theta))G(r\cos\theta, r\sin\theta)\big)\sin\theta,\\ \frac{d\theta}{dt} &= -1 + \frac{\varepsilon}{r}\Big(\big(H(r\cos\theta, r\sin\theta) - \alpha\,\mathrm{sgn}(\sin(2\theta))G(r\cos\theta, r\sin\theta)\big)\cos\theta\Big).\end{aligned}$$

Taking as new independent variable the angle θ this discontinuous differential system becomes

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon\big(\alpha\,\mathrm{sgn}(\sin(2\theta))G(r\cos\theta, r\sin\theta) - H(r\cos\theta, r\sin\theta)\big)\sin\theta + O(\varepsilon^2)\\ &= \varepsilon F(\theta, r) + O(\varepsilon^2).\end{aligned} \tag{12}$$

Applying Theorem 7 to this discontinuous differential equation, where the variables of this theorem are in our case $t = \theta$, $T = 2\pi$, $\mathbf{x} = r$, $\mathcal{M} = h^{-1}(0) = \{xy = 0\}$, we compute

$$f(r) = \int_0^{2\pi} F(\theta, r)d\theta = f_2(r),$$

where $f_2(r)$ is the function defined in the statement of Proposition 2. Since $f_2(r)$ is C^1, if r^* is a simple zero of $f_2(r)$, then the Brouwer degree $d_B(f_2, V, r^*) \neq 0$ being V a convenient open neighborhood of r^*. Therefore, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (12) has a periodic solution $r(\theta, \varepsilon)$ such that $r(0, \varepsilon) \to r^*$ when $\varepsilon \to 0$. Going back through the polar change of variables we obtain that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (6) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (r^*, 0)$ when $\varepsilon \to 0$. This completes the proof of the proposition. $\square$

Proof of Proposition 3. The discontinuous differential system (7) is already in the form (13) for applying the averaging theory described in Theorem 7, where now the variables of Theorem 7 are $t = \theta$, $T = 2\pi$, $\mathbf{x} = (x, y)$, $\mathcal{M} = h^{-1}(0) = \{x = 0\}$, $F(t, \mathbf{x}) = F(\theta, x, y) = (F_1(\theta, x, y), F_2(\theta, x, y))$ where

$$\begin{aligned}F_1(\theta, x, y) &= y,\\ F_2(\theta, x, y) &= \alpha\,\mathrm{sign}(x)G(\theta, x, y) + H(\theta, x, y).\end{aligned}$$

Therefore we apply Theorem 7 to the discontinuous differential system (7) and we obtain

$$f(x, y) = \int_0^{2\pi} F(\theta, x, y)d\theta,$$

where $f(x, y) = (g_1(x, y), g_2(x, y))$ with

$$\begin{aligned}g_1(x, y) &= y,\\ g_2(x, y) &= \int_0^{2\pi} H(\theta, x, y)\,d\theta + \alpha\left(\int_{\pi/2}^{3\pi/2} G(\theta, x, y)\,d\theta - \int_{-\pi/2}^{\pi/2} G(\theta, x, y)\,d\theta\right).\end{aligned}$$

A solution (x^*, y^*) of the system $g_1(x,y) = g_2(x,y) = 0$ satisfies $y^* = 0$ and x^* is a solution of $f_3(x) = 0$ where this function is the one defined in the statement of Proposition 3. Since G and H are C^1 functions in their two variables, it follows that $g_1(x,y)$, $g_2(x,y)$ and $f_3(x)$ are C^1. Consequently, if $(x^*, 0)$ is a zero of the system $g_1(x,y) = g_2(x,y) = 0$, and the Jacobian

$$\det \left(\begin{array}{cc} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{array} \right) \Bigg|_{(x,y)=(x^*,0)} = \left. \frac{df_3}{dx} \right|_{x=x^*} \neq 0,$$

then the Brouwer degree $d_B(f, V, (x^*, 0)) \neq 0$ being V a convenient open neighborhood of $(x^*, 0)$, see again for more details on the Brouwer degree [3] and [9]. Hence, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (7) has a periodic solution $(x(\theta, \varepsilon), y(\theta, \varepsilon))$ such that $(x(0, \varepsilon), y(0, \varepsilon)) \to (x^*, 0)$ when $\varepsilon \to 0$. So, the proposition follows. $\square$

Proof of Proposition 4. The discontinuous differential system (8) is in the form (13) for applying the averaging theory described in Theorem 7, where the variables of Theorem 7 now are $t = \theta$, $T = 2\pi$, $\mathbf{x} = (x, y)$, $\mathcal{M} = h^{-1}(0) = \{xy = 0\}$, $F(t, \mathbf{x}) = F(\theta, x, y) = (F_1(\theta, x, y), F_2(\theta, x, y))$ where

$$\begin{aligned} F_1(\theta, x, y) &= y, \\ F_2(\theta, x, y) &= \alpha \operatorname{sign}(xy) G(\theta, x, y) + H(\theta, x, y). \end{aligned}$$

By applying Theorem 7 to the discontinuous differential system (8) and we obtain

$$f(x, y) = \int_0^{2\pi} F(\theta, x, y) d\theta,$$

where $f(x, y) = (g_1(x, y), g_2(x, y))$ with

$$\begin{aligned} g_1(x, y) &= y, \\ g_2(x, y) &= \int_0^{2\pi} H(\theta, x, y)\, d\theta - \alpha \left(\int_0^{\pi/2} G(\theta, x, y)\, d\theta + \int_\pi^{3\pi/2} G(\theta, x, y)\, d\theta \right) \\ &\quad + \alpha \left(\int_{\pi/2}^{\pi} G(\theta, x, y)\, d\theta + \int_{3\pi/2}^{2\pi} G(\theta, x, y)\, d\theta \right). \end{aligned}$$

A solution (x^*, y^*) of the system $g_1(x,y) = g_2(x,y) = 0$ satisfies $y^* = 0$ and x^* is a solution of $f_4(x) = 0$ where this function is the one defined in the statement of Proposition 4. Since $g_1(x,y)$, $g_2(x,y)$ and $f_4(x)$ are C^1, and if $(x^*, 0)$ is a zero of the system $g_1(x,y) = g_2(x,y) = 0$, then the Jacobian

$$\det \left(\begin{array}{cc} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{array} \right) \Bigg|_{(x,y)=(x^*,0)} = \left. \frac{df_4}{dx} \right|_{x=x^*} \neq 0,$$

then the Brouwer degree $d_B(f, V, (x^*, 0)) \neq 0$ being V a convenient open neighborhood of $(x^*, 0)$. Therefore, by Theorem 7 it follows that for $\varepsilon \neq 0$ sufficiently small the discontinuous differential system (8) has a periodic solution $(x(\theta, \varepsilon), y(\theta, \varepsilon))$

such that $(x(0,\varepsilon), y(0,\varepsilon)) \to (x^*, 0)$ when $\varepsilon \to 0$. In short, the proposition is proved. $\square$

3. Proof of the applications

Here we prove the two corollaries.

Proof of Corollary 5. The autonomous discontinuous differential equation of second order (9) is a particular case of equation (1) with

$$G(\theta, u, u') = uu' \quad \text{and} \quad H(\theta, u, u') = \frac{\alpha}{\pi} u'.$$

Then computing for equation (9) the function $f_1(r)$ given in the statement of Proposition 1 we get

$$f_1(r) = -\frac{\alpha}{3} r(4r - 3).$$

Hence, $f_1(r) = 0$ has a unique positive simple root $r = 3/4$. Going back through the changes of variables described in the proof of Proposition 1, we obtain the result stated in the corollary. $\square$

Proof of Corollary 6. The non–autonomous discontinuous differential equation of second order (10) is a particular case of equation (3) with

$$G(\theta, r, r') = (2 - 3r) \cos\frac{\theta}{2} \quad \text{and} \quad H(\theta, r, r') = -\frac{\sqrt{2}\,\alpha}{\pi} r^2.$$

Then computing for equation (10) the function $f_3(x)$ given in the statement of Proposition 3 we get

$$f_3(x) = -2\sqrt{2}\,\alpha(x - 2)(x - 1).$$

Therefore, $f_3(x) = 0$ has two simple roots $x = 1$ and $x = 2$. Going back through the changes of variables described in the proof of Proposition 3, it follows the result stated in the corollary. $\square$

Appendix: Averaging theory of first order for discontinuous differential systems

We need the following recent result of [8] on averaging theory for computing periodic orbits of discontinuous differential systems. Its proof uses the theory on the Brouwer degree $d_B(f, V, 0)$ for finite dimensional spaces (see the appendix A of [8] for a definition of the Brouwer degree), and it is based on the averaging theory for continuous non–smooth differential system stated in [4].

Theorem 7. *We consider the following discontinuous differential system*

$$\mathbf{x}'(t) = \varepsilon F(t, \mathbf{x}) + \varepsilon^2 R(t, \mathbf{x}, \varepsilon), \tag{13}$$

with

$$F(t, \mathbf{x}) = F_1(t, \mathbf{x}) + \operatorname{sign}(h(t, \mathbf{x})) F_2(t, \mathbf{x}),$$
$$R(t, \mathbf{x}, \varepsilon) = R_1(t, \mathbf{x}, \varepsilon) + \operatorname{sign}(h(t, \mathbf{x})) R_2(t, \mathbf{x}, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having 0 as a regular value. Denote by $\mathcal{M} = h^{-1}(0)$, by $\Sigma = \{0\} \times D \not\subseteq \mathcal{M}$, by $\Sigma_0 = \Sigma \backslash \mathcal{M} \neq \varnothing$, and its elements by $z \equiv (0, z) \notin \mathcal{M}$.

Define the averaged function $f : D \to \mathbb{R}^n$ *as*

$$f(\mathbf{x}) = \int_0^T F(t, \mathbf{x})dt. \tag{14}$$

We assume the following three conditions.

(i) F_1, F_2, R_1, R_2 *and* h *are locally* L*–Lipschitz with respect to* $\mathbf{x}$*;*

(ii) *for* $a \in \Sigma_0$ *with* $f(a) = 0$*, there exist a neighbourhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \backslash \{a\}$ *and* $d_B(f, V, 0) \neq 0$*.*

(iii) *If* $\partial h/\partial t(t_0, z_0) = 0$ *for some* $(t_0, z_0) \in \mathcal{M}$*, then* $\big(\langle \nabla_{\mathbf{x}} h, F_1\rangle^2 - \langle \nabla_{\mathbf{x}} h, F_2\rangle^2\big)(t_0, z_0) > 0$*.*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists a* T*–periodic solution* $\mathbf{x}(\cdot, \varepsilon)$ *of system* (13) *such that* $\mathbf{x}(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$*.*

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References

[1] E.A. Barbashin, *Introduction to the theory of stability*, Translated from the Russian by Transcripta Service, London. Edited by T. Lukes Wolters-Noordhoff Publishing, Groningen, 1970.

[2] M. di Bernardo et al., *Bifurcations in non–smooth dynamical systems*, Publications of the Bristol Centre for Applied Nonlinear Mathematics, vol. **4**, 2005.

[3] F. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39.

[4] A. Buica and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bulletin des Sciences Mathemàtiques **128** (2004),7–22.

[5] J.A. Cid and L. Sanchez, *Periodic solutions for second order differential equations with discontinuous restoring forces*, J. Math. Anal. Appl. **288** (2003), 349–364.

[6] A. Jacquemard and M.A. Teixeira, *Periodic solutions of a class of non–autonomous second order differential equations with discontinuous right–hand side*, Physica D **241** (2012), 2003–2009.

[7] M. Kunze, T. Küpper and J. You, *On the application of KAM theory to discontinuous dynamical systems*, J. Differential Equations **139** (1997), 1–21.

[8] J. Llibre, D.D. Novaes and M.A. Teixeira, *Averaging methods for studying the periodic orbits of discontinuous differential systems*, http://arxiv.org/abs/1205.4211.

[9] N.G. Lloyd, *Degree Theory*, Cambridge University Press, 1978.

[10] O. Makarenkov and J.S.W. Lamb, *Dynamics and bifurcations of nonsmooth systems: a survey*, Physica D **241** (2012), 1826–1844.

[11] O. Makarenkov and P. Nistri, *Periodic solutions for planar autonomous systems with nonsmooth periodic perturbations*, J. Math. Anal. Appl. **338** (2008), 1401–1417.

[12] N.Minorsky, *Nonlinear oscillations*, D. Van Nostrand Company, Inc., Princeton, 1962.

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